Answers to some exercises

Chapter 1

Section 1.5

1 **a)**

$s1 := 0$; $s2 := 0$;
for $k := 1, 2, \ldots, n$
  if $a_k > 0$
    $s1 := s1 + a_k$;
  else
    $s2 := s2 + a_k$;
$s2 := -s2$;

Note that we could also replace the statement in the **else**-branch by $s2 := s2 - a_k$ and leave out the last statement.

b) We introduce two new variables *pos* and *neg* which count the number of positive and negative elements, respectively.

$s1 := 0$; $pos := 0$;
$s2 := 0$; $neg := 0$;
for $k := 1, 2, \ldots, n$
  if $a_k > 0$
    $s1 := s1 + a_k$;
    $pos := pos + 1$;
  else
    $s2 := s2 + a_k$;
    $neg := neg + 1$;
$s2 := -s2$;

2 We represent the three-digit numbers by their decimal numerals which are integers in the range 0–9. The numerals of the number $x = 431$ for example, is represented by $x_1 = 1$, $x_2 = 3$ and $x_3 = 4$. Adding two arbitrary such numbers x and y produces a sum z which can be computed by the algorithm

if $x_1 + y_1 < 10$
  $z_1 := x_1 + y_1$;
else
  $x_2 := x_2 + 1$;

$z_1 := x_1 + y_1 - 10;$
if $x_2 + y_2 < 10$
$z_2 := x_2 + y_2;$
else
$x_3 := x_3 + 1;$
$z_2 := x_2 + y_2 - 10;$
if $x_3 + y_3 < 10$
$z_3 := x_3 + y_3;$
else
$z_4 := 1;$
$z_3 := x_3 + y_3 - 10;$

3 We use the same representation as in the solution for exercise 2. Multiplication of two three-digit numbers x and y can then be performed by the formulas

$$product1 := x_1 * y1 + 10 * x_1 * y_2 + 100 * x_1 * y_3;$$
$$product2 := 10 * x_2 * y1 + 100 * x_2 * y_2 + 1000 * x_2 * y_3;$$
$$product3 := 100 * x_3 * y1 + 1000 * x_3 * y_2 + 10000 * x_3 * y_3;$$
$$product := product1 + product2 + product3;$$

Chapter 2

Section 2.3

1 The truth table is

p	q	r	$p \oplus q$	$(p \oplus q) \oplus r$	$q \oplus r$	$p \oplus (q \oplus r)$
F	F	F	F	F	F	F
F	F	T	F	T	T	T
F	T	F	T	T	T	T
F	T	T	T	F	F	F
T	F	F	T	T	F	T
T	F	T	T	F	T	F
T	T	F	F	F	T	F
T	T	T	F	T	F	T

2 Solution by truth table for $\neg(p \wedge q) = \neg(p \vee q)$

p	q	$p \wedge q$	$\neg p$	$\neg q$	$\neg(p \wedge q)$	$(\neg p) \vee (\neg q)$
F	F	F	T	T	T	T
F	T	F	T	F	T	T
T	F	F	F	T	T	T
T	T	T	F	F	F	F

Solution by truth table for $\neg(p \vee q) = \neg(p \wedge q)$

p	q	$p \vee q$	$\neg p$	$\neg q$	$\neg(p \vee q)$	$(\neg p) \wedge (\neg q)$
F	F	F	T	T	T	T
F	T	T	T	F	F	F
T	F	T	F	T	F	F
T	T	T	F	F	F	F

3 No answer given.

4 No answer given.

Chapter 3

Section 3.2

1
- **a)** 220
- **b)** 32
- **c)** 10001
- **d)** 1022634
- **e)** 123456
- **f)** $7e$

2
- **a)** 131
- **b)** 67
- **c)** 252

3
- **a)** 100100
- **b)** 1000000
- **c)** 11010111

4
- **a)** $4d$
- **b)** c
- **c)** $29e4$
- **d)** 0.594
- **e)** 0.052
- **f)** $0.ff8$

5
- **a)** 111100
- **b)** 100000000
- **c)** 111001010001
- **d)** 0.000010101010
- **e)** 0.000000000001
- **f)** 0.111100000001

6
- **a)** $7 = 10_7$, $37 = 10_{37}$ and $4 = 10_4$
- **b)** $\beta = 13, \beta = 100$

7
- **a)** $400 = 100_{20}$, $4 = 100_2$ and $278 = 100_{17}$
- **b)** $\beta = 5, \beta = 29$

Section 3.3

1
- **a)** 0.01
- **b)** 0.102120102120102120…
- **c)** 0.01
- **d)** 0.001111111…
- **e)** 0.7
- **f)** 0.6060606…
- **g)** $0.e$
- **h)** 0.24
- **i)** 0.343

2 $\pi_9 \approx 3.12_9$

3 No answer given.

4 $c-1$

5 No answer given.

6 No answer given.

Section 3.4

1
- **a)** 4_7
- **b)** 13_6
- **c)** 10001_2
- **d)** 1100_3
- **e)** 103_5
- **f)** $4_5 = 4_7$

2
- **a)** 3_8
- **b)** 11_2
- **c)** 174_8
- **d)** 112_3
- **e)** 24_5
- **f)** -5_7

3
- **a)** 1100_2
- **b)** 10010_2
- **c)** 1210_3
- **d)** 141_5

e) 13620_8

f) 10220_3

g) 1111_2

Chapter 4

Section 4.2

1 Largest integer: $7fffffff_{16}$.

Smallest integer: 80000000_{16}.

2 See Internet

3 Answers to some of the questions:

- **a)** 0.4752735×10^7
- **b)** $0.602214179 \times 10^{24}$
- **c)** 0.8617343×10^{-4}.

4 $0.1001\ 1100\ 1111\ 0101\ 1010 \ldots \times 2^4$

Section 4.3

1
- **a)** $0101\ 1010_2 = 5a_{16}$
- **b)** $1100\ 0011\ 1011\ 0101_2 = c3b5_{16}$
- **c)** $1100\ 1111\ 1011\ 1000_2 = cfb8_{16}$
- **d)** $1110\ 1000\ 1011\ 1100\ 1011\ 0111_2 = e8bcb7_{16}$

2
- **a)** $0000\ 0000\ 0101\ 1010_2 = 005a_{16}$
- **b)** $0000\ 00001111\ 0101_2 = 00f5_{16}$
- **c)** $0000\ 0011\ 1111\ 1000_2 = 03f8_{16}$
- **d)** $1000\ 1111\ 0011\ 0111_2 = 8f37_{16}$

3
- **a)** ï¿½: Ãï¿½, ï¿½: Ã¸, ï¿½: Ã¥
- **b)** Nothing or error message; these codes are not valid UTF-8 codes
- **c)** ï¿½: NULï¿½, ï¿½: NULï¿½, ï¿½: NULï¿½; each character is preceded (or followed for LE) by a trailing null character, this has no visible impact on the displayed text. The opposite again yields illegitimate UTF-16 encodings (too short).
- **d)** The conversion from UTF-8 to UTF-16 yields the following Hangul symbols:

  ï¿½: 쎻, ï¿½: 쎡, ï¿½: 쎶

  The conversion from UTF-16 to UTF-8 yields illegitimate codes, though there will be an allowed null character preceding (or following for LE) each prohibited letter.

4 No answer given.

5 No answer given.

6 No answer given.

7 No answer given.

8 No answer given.

9 No answer given.

10 No answer given.

Chapter 5

Section 5.2

1 No answer given.

2 No answer given.

3 No answer given.

4 **a)** 0.1647×10^2

b) 0.1228×10^2

c) 0.4100×10^{-1}

d) 0.6000×10^{-1}

e) -0.5000×10^{-2}

5 **a)** Normalised number in base β: A nonzero number a is written as

$$a = \alpha \times \beta^n$$

where $\beta^{-1} \leq |\alpha| < 1$.

b) In any numeral system we have three cases to consider when defining rounding rules. Note also that it is sufficient to define rounding for two-digit fractional numbers.

In the octal numeral system the three rules are:

1. A number $(0.d_1 d_2)_8$ is rounded to $0.d_1$ if the digit d_2 is 0, 1, 2 or 3.
2. If $d_1 < 7$ and d_2 is 4, 5, 6, or 7, then $(0.d_1 d_2)_8$ is rounded to $0.\tilde{d}_1$ where $\tilde{d}_1 = d_1 + 1$.
3. A number $(0.7d_2)_8$ is rounded to 1.0 if d_2 is 4, 5, 6, or 7.

c) No answer given.

6 One possible program:

```
n := 1;
while 1.0 + 2^(-n) > 1.0
    n := n + 1;
print n;
```

7 No answer given.

8 No answer given.

Section 5.3

1 Relative errors:

- **a)** $r = 0.0006$
- **b)** $r \approx 0.0183$
- **c)** $r \approx 2.7 \times 10^{-4}$
- **d)** $r \approx 0.94$

2 No answer given.

3 No answer given.

4 No answer given.

Section 5.4

1

- **a)** No answer given.
- **b)** The fomula $\ln x^2 - \ln(x^2 + x)$ is problematic for large values of x since then the two logarithms will become almost equal and we get cancellation. Using properties of the logarithm, the expression can be rewritten as

$$\ln x^2 - \ln(x^2 + x) = \ln\left(\frac{x^2}{x^2 + x}\right) = \ln\left(\frac{x}{x+1}\right)$$

  which will not cause problems with cancellation.
- **c)** No answer given.

2 No answers given

3 No answer given.

4 No answer given.

5 No answer given.

Chapter 6

Section 6.1

1 In simpler English the riddle says: Diophantus' youth lasted 1/6 of his life. He had the first beard in the next 1/12 of his life. At the end of the following 1/7 of his life Diophantus got married. Five years laater his son was born. His son lived exactly 1/2 of Diophantus' life. Diophantus died 4 years after the death of his son. Solution: If d and s are the ages of Diophantus and his son when they died, then the epitaph corresponds to the two equations

$$d = (1/6 + 1/12 + 1/7)d + 5 + s + 4,$$
$$s = 1/2d.$$

If we solve these we obtain $s = 42$ years and $d = 84$ years.

Section 6.2

1 **a)** $x_2 = 2, x_3 = 5, x_4 = 13, x_5 = 34$
b) $x_2 = 17, x_3 = 32, x_4 = 83, x_5 = 179$
c) $x_2 = 4, x_3 = 16, x_4 = 128, x_5 = 4096$
d) No answer given.

2 **a)** Linear.
b) Nonlinear.
c) Nonlinear.
d) Linear.

Section 6.4

1 **a)** $x_n = 3^n \cdot \frac{5}{3}$
b) No answer given.
c) $x_n = (1 - 2n)(-1)^n$
d) $x_n = \frac{3}{4} \cdot 3^n + \frac{5}{4}(8 - 1)^n$

2 No answer given.

3 No answers given.

4 No answers given.

Section 6.5

1 **a)** $x_n = 3 - 3^{-n}$.
b) $x_n = 1/7$.
c) $x_n = (2/3)^n$.

2 No answers given.

3 No answers given.

4 **a)** Solution determined by the initial conditions: $x_n = 15^{-n}$.
b) No answer given.
c) $n \approx 24$.
d) No answer given.

5 **a)** Solution determined by the initial conditions: $x_n = 2^{-n}$.
b) No answer given.
c) No answer given.

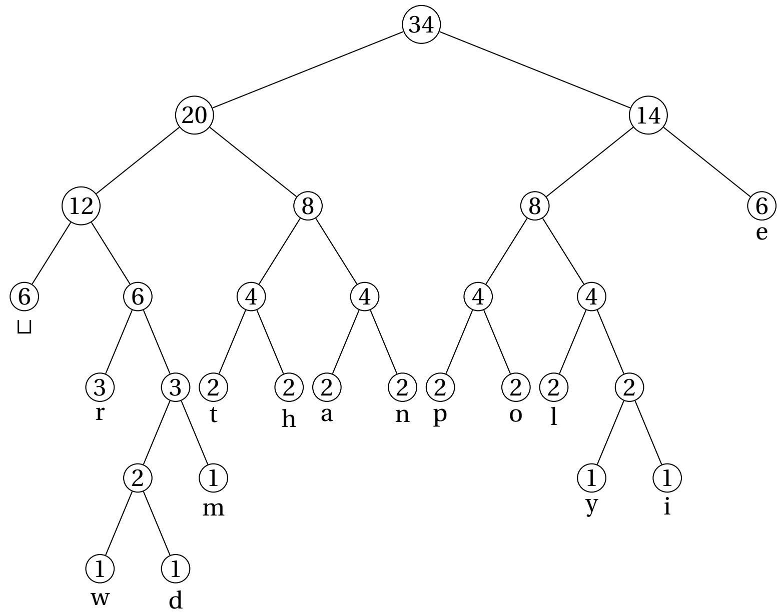

Figure 16. The Huffman tree for the text 'there are many people in the world'.

Chapter 7

Section 7.1

1 No answers given.

Section 7.2

1 Huffman coding of the text "There are many peopole in the world". (In this soultion we will treat the capital t in "There" as if it were not capitalized.)

a)

$$\begin{array}{ccc} f(t)=2, & f(a)=2, & f(o)=2, \\ f(h)=2, & f(m)=1, & f(l)=2, \\ f(e)=6, & f(n)=2, & f(i)=1, \\ f(r)=3, & f(y)=1, & f(w)=1, \\ f(\sqcup)=6, & f(p)=2, & f(d)=1. \end{array}$$

b) An example of a Huffman tree for this text can be seen in figure 16:

c) The Huffman coding for the text "there are many people in the world" is then:

0100 0101 11 0010 11 000 0110 0010 11 000
00111 0110 0111 10110 000 1000 11 1001 1000
1010 11 000 10111 0111 000 0100 0101 11 000
001100 1001 0010 1010 001101

The entropy is:

$$H = 3.6325 \quad (.6)$$

which means an optimal coding of the text would use 3.6325 bits per symbol. There are 34 symbols so the minimum coding would consist of 15 bytes and 4 bits. The Huffman coding above gave 15 bytes and 5 bits of information, so this coding is very good.

2 **a)** Use ternary trees instead of binary ones. (Each tree has either zero or three subtrees/children).

b) Use n-nary trees. (Each tree has either zero or n subtrees/children)

3 **a)**

$$f(A) = 4,$$
$$f(B) = 2,$$
$$f(C) = 2,$$

One of the four possible Huffman codings are:

0 10 0 11 0 10 11 0

The entropy is

$$H = 1.5 \quad (.7)$$

This gives an optimal coding with 12 bits for 8 symbols, which is just what the Huffman coding gave.

b) Dividing all the frequensies by 2 and interchanging A with C in the four trees in *a)* gives the four trees for this problem.The four sets of codes are the same (with A interchanged by C) and so is the entropy so the situation is still optimal.

4 Frequensies used are all 1.

Section 7.3

1 $\log_2 x = \ln x / \ln 2$.

2 No answers given.

3 No answers given.

Section 7.4

1 **a)**

$$f(A) = 9, \qquad p(A) = 0.1,$$
$$f(B) = 1, \qquad p(B) = 0.9,$$

b) 5 bits

c) 01110

2 **a)** $H = 2$

b) 2 bits per symbol

c) 2m + 1 bits $\frac{2m+1}{m} \approx 2$ bits per symbol

d)

00 10 11 01 00 10

e)

00 10 11 01 00 10 1

3

BCBBCBBBCB

4

01 01 11 10 00

5

$$f(x) = c + (y - a)\frac{d - c}{b - a} \tag{.8}$$

Section 7.6

1 No answer given.

Chapter 8

Chapter 9

Section 9.1

1 No answer given.

2 **a)** $T_2(x;1) = 1 - 3x + 3x^2$.
b) $T_2(x;0) = 12x^2 + 3x + 1$.
c) $T_2(x;0) = 1 + x\ln 2 + (\ln 2)^2 x^2/2$.

3 No answer given.

4 No answer given.

Section 9.2

1 **a)**

$$p_3(x) = -\frac{(x-1)(x-3)(x-4)}{12} - \frac{x(x-1)(x-4)}{3} + \frac{x(x-1)(x-3)}{12}.$$

b) No answer given.
c)

$$p_3(x) = 1 - x + \frac{2}{3}x(x-1) - \frac{1}{3}x(x-1)(x-3).$$

Section 9.3

1 **a)** $f[0,1,2,3] = 0$.
b) No answer given.

2 **a)** The Newton form is

$$p_2(x) = 2 - x.$$

b) No answer given.

3 **a)** Linear interpolant p_1:

$$p_1(x) = y_1 + (y_2 - y_1)(x-1).$$

Error at x:

$$f[1,2,x](x-1)(x-2) = \frac{f''(\xi)}{2}(x-1)(x-2)$$

where ξ is a number in the smallest interval (a,b) that contains all of 1, 2, and x.
Error at $x = 3/2$:

$$\frac{f''(\xi_1)}{8}$$

where ξ is a number in the interval $(1,2)$.

b) Cubic interpolant:

$$p_3(x) = y_0 + (y_1 - y_0)x + \frac{y_2 - 2y_1 + y_0}{2} x(x-1) + \frac{y_3 - 3y_2 + 3y_1 - y_0}{6} x(x-1)(x-2).$$

Error:

$$f[0,1,2,3,x]x(x-1)(x-2)(x-3) = \frac{f^{(iv)}(\xi)}{4!} x(x-1)(x-2)(x-3)$$

where ξ is now a number in the smallest open interval that contains all of 0, 1, 2, 3, and x. With $x = 3/2$ this becomes

$$\frac{3}{128} f^{(iv)}(\xi_3)$$

where ξ_3 is a number in the interval $(0,3)$.

Section 9.4

1 No answer given.

Chapter 10

Section 10.2

1 **a)** Approximation after 10 steps: 0.73876953125.

b) To get 10 correct digits it is common to demand that the relative error is smaller than 5×10^{-11}, even though this does not always ensure that we have 10 correct digits. A challenge with the relative error is that it requires us to know the exact zero. In our case we have a very good approximation that we could use, but as we commented when we discussed properties of the relative error, it is sufficient to use a rough estimate, like 0.7 in this case. The required inequality is therefore

$$\frac{1}{2^N 0.7} \leq 5 \times 10^{-11}.$$

This inequality can be easily solved and leads to $N \geq 35$.

c) Actual error: 1.3×10^{-11}

d) No answer given.

2 No answers given.

3 No answers given.

4 No answers given.

Section 10.3

1 **a)** $f(x) = x^2 - 3$. One iteration gives the approximation 1.6666666666666667 which has two correct digits ($\sqrt{3} \approx 1.7320508075688772935$ with 20 correct digits). After 6 iterations we obtain the approximation 1.732050807568877.

b) $f(x) = x^{12} - 2$.

c) $f(x) = \ln x - 1$.

2 No answer given.

3 No answers given.

Section 10.4

1 If you do the computations with 64-bit floating-point numbers, you have full machine accuracy after just 4 iterations. If you do 7 iterations you actually have about 164 correct digits.

2
- **a)** Midpoint after 10 iterations: 3.1416015625.
- **b)** Approximation after 4 iterations: 3.14159265358979.
- **c)** Approximation after 4 iterations: 3.14159265358979.
- **d)** No answer given.

3 No answer given.

4
- **a)** No answer given.
- **b)** $e_{n+1} = e_{n-1}e_n\big/(x_{n-1} + x_n)$, where $e_n = \sqrt{2} - x_n$.

5
- **a)** No answer given
- **b)** After 5 iterations we have the approximation 0.142857142857143 in which all the digits are correct (the fourth approximation has approximate error 6×10^{-10}).

6
- **a)** No answer given
- **b)** No answer given
- **c)** An example where $x_n > c$ for $n > 0$ is $f(x) = x^2 - 2$ with $c = \sqrt{2}$ (choose for example $x_0 = 1$). If we use the same equation, but choose $x_0 = -1$, we converge to $-\sqrt{2}$ and have $x_n < c$ for large n (in fact $n > 0$).

  An example where the iterations jump around is in computing an approximation to a zero of $f(x) = \sin x$, for example with $x_0 = 4$ (convergence to $c = \pi$).

Chapter 11

Section 11.1

1 **a)** No answer given.

b) $h^* \approx 8.4 \times 10^{-9}$.

2 No answers given.

Section 11.2

1 $f'(a) \approx p_2'(a) = -(f(a+2h) - 4f(a+h) + 3f(a))/(2h)$.

Section 11.3

1 **a)** No answer given.

b) $h^* \approx 5.9 \times 10^{-6}$.

2 **a)** No answer given.

b) With 6 digits:
$(f(a+h) - f(a))/h = 0.455902$, relative error: 0.0440981.
$(f(a) - f(a-h))/h = 0.542432$, relative error: 0.0424323.
$(f(a+h) - f(a-h))/(2h) = 0.499167$, relative error: 0.000832917.

c) No answer given.

d) No answer given.

e) No answer given.

f) No answer given.

3 No answer given.

4 **a)** No answer given.

b) No answer given.

c) With 6 digits:
$(f(a+h) - f(a))/h = 0.975$, relative error: 0.025.
$(f(a) - f(a-h))/h = 1.025$, relative error: 0.025.
$(f(a+h) - f(a-h))/(2h) = 1$, relative error: 8.88178×10^{-16}.

5 **a)** Optimal h: 2.9×10^{-6}.

b) Optimal h: 3.3×10^{-6}

Section 11.4

1 **a)** No answer given.

b) Opitmal h: 9.9×10^{-4}.

2 No answer given.

Section 11.5

1 **a)** No answer given.

b) Optimal h: 2.24×10^{-4}.

2 No answer given.

3 **a)** $c_1 = -1/(2h)$, $c_2 = 1/(2h)$.

b) No answer given.

c) $c_1 = -1/h^2$, $c_2 = 2/h^2$, $c_3 = -1/h^2$.

d) No answer given.

Chapter 12

Section 12.1

1 **a)** $\underline{I} \approx 1.63378$, $\overline{I} \approx 1.805628$.

b) $\left|I - \underline{I}\right| \approx 0.085$, $\frac{|I-\underline{I}|}{|I|} = 0.0491781$.

$\left|I - \overline{I}\right| \approx 0.087$, $\frac{\left|I-\overline{I}\right|}{|I|} = 0.051$.

c) No answer given.

Section 12.2

1 Approximation: 0.530624 (with 6 digits).

2 **a)** Approximation with 10 subintervals: 1.71757 (with 6 digits).

b) $h \leq 2.97 \times 10^{-5}$.

3 **a)** Approximation with 10 subintervals: 5.36648 (with 6 digits).

b) $h \leq 4.89 \times 10^{-5}$.

4 No answer given.

Section 12.3

1 Approximation: 0.519725 (with 6 digits).

2 **a)** Approximation with 10 subintervals: 1.71971 (with 6 digits).

b) $h \leq 1.48 \times 10^{-5}$.

3 No answer given.

4 No answer given.

5 No answer given.

Section 12.4

1 Approximation: 0.527217 (with 6 digits).

2 **a)** 115 471 evaluations.
b) 57 736 evaluations.
c) 192 evaluations.

3 **a)** Approximation with 10 subintervals: 1.718282782 (with 10 digits).
b) $h \le 1.8 \times 10^{-2}$.

4 No answers given.

5 $w_1 = w_3 = (b-a)/6$, $w_2 = 2(b-a)/3$.

Chapter 13

Chapter 14

Section 14.1

1 **a)** Linear.
b) Nonlinear.
c) Nonlinear.
d) Nonlinear.
e) Linear.

Section 14.2

1 The general solution is $x(t) = 1 + Ce^{\cos t}$.

2 **a)** $x(t) = 1$ will cause problems.
b) The differential equation is not defined for $t = 1$.
c) The equation is not defined when $x(t)$ is negative.
d) The equation does not hold if $x'(t) = 0$ or $x(t) = 0$ for some t.
e) The equation is not defined for $|x(t)| > 1$.
f) The equation is not defined for $|x(t)| > 1$.

Section 14.3

1 **a)** $x(0.3) \approx 1.362$.
b) $x(0.3) \approx 0.297517$.
c) $x(0.3) \approx 1.01495$.
d) $x(1.3) \approx 1.27488$.
e) $x(0.3) \approx 0.297489$.

2 No answer given.

3 No answer given.

4 No answer given.

Section 14.4

1 If the step length is h, we obtain the approximation

$$x(h) \approx x(0) + hf(t,x) = 1 + h\sin h.$$

The error is given by

$$R_1(h) = \frac{h^2}{2}x''(\xi)$$

where $\xi \in (0,h)$. Since $x'(t) = \sin x(t)$, we have

$$x''(t) = x'(t)\cos x(t) = \sin x(t)\cos x(t) = \frac{\sin\bigl(2x(t)\bigr)}{2}$$

We therefore have $\bigl|x''(t)\bigr| \leq 1/2$, so

$$\bigl|R_1(h)\bigr| \leq \frac{h^2}{4}.$$

Section 14.5

1 **a)** $x''(0) = 1$, $x'''(0) = 1$.

b) $x''(0) = 1$, $x'''(0) = 0$.

c) $x''(1) = 0$, $x'''(0) = 0$.

d) $x''(1) = 0$, $x'''(1) = 0$.

Section 14.6

1 **a)** Euler: $x(1) \approx 5.01563$.
Quadratic Taylor: $x(t) \approx 5.05469$.
Quartic Taylor: $x(t) \approx 5.14583$.

b) Euler: $x(1) \approx 2.5$.
Quadratic Taylor: $x(t) \approx 3.28125$.
Quartic Taylor: $x(t) \approx 3.43469$.

c) Euler: $x(1) \approx 12.6366$.
Quadratic Taylor: $x(t) \approx 13.7823$.
Quartic Taylor: $x(t) \approx 13.7102$.

2 **a)** Euler: $x(0.5) \approx 1.5$.
Since we only take one step, Euler's method is just the approximation

$$x(h) \approx x(0) + hx'(0)$$

where $h = 0.5$, $x(0) = 1$, and $x'(t) = e^{-t^2}$. The error is therefore given by the remainder in Taylor's formula

$$R_1(h) = \frac{h^2}{2} x''(\xi_1),$$

where $\xi_1 \in (0, h)$. Since the right-hand side

$$g(t) = e^{-t^2}$$

of the differential equation is independent of x, we simply have

$$x''(t) = \frac{d}{dt}\big(x'(t)\big) = \frac{d}{dt}\big(g(t)\big) = \frac{d}{dt}\Big(e^{-t^2}\Big) = -2te^{-t^2}.$$

To bound the absolute error $|R_1(h)|$, we therefore need to bound the absolute value of this expression. A simple upper bound is obtained by using the estimates $|t| \le 0.5$ and $e^{-t^2} \le 1$,

$$|R_1(0.5)| \le \frac{0.5^2}{2} 0.5 = \frac{1}{16} = 0.0625.$$

The actual error turns out to be about 0.039.

b) Quadratic Taylor: $x(0.5) \approx 1.5$.
In this case we need to estimate $R_2(0.5)$, where

$$R_2(h) = \frac{h^3}{6} x'''(\xi_2)$$

and $\xi_2 \in (0, h)$. We have $x'''(t) = g''(t) = 2(2t^2 - 1)e^{-t^2}$. The maximum of the first factor is 2 on the interval $[0, 0.5]$ and the maximum of the second factor is 1. We therefore have

$$|R_2(0.5)| \le 2\frac{0.5^3}{6} \approx 0.042.$$

c) Cubic Taylor: $x(0.5) \approx 1.458333$.
In this case the remainder is

$$R_3(h) = \frac{h^4}{24} x''''(\xi_3),$$

where $\xi_3 \in (0, h)$ and $x''''(t) = g'''(t) = 4t(3 - 2t^2)e^{-t^2}$. The quick estimate is

$$4t \le 2, \quad 3 - 2t^2 \le 3, \quad e^{-t^2} \le 1$$

which leads to

$$|R_3(0.5)| \le \frac{0.5^4}{24} \times 3 \times 2 = \frac{0.5^4}{4} \approx 0.016.$$

The true error is approximately 0.0029.

We can improve the estimate slightly by finding the maximum of $g'''(t)$. On the interval $[0, 0.5]$ this is an increasing function so its maximum is $g'''(0.5) \approx 3.89 \le 4$. This leads to the slightly better estimate

$$|R_3(0.5)| \le \frac{0.5^4}{24} 4 \approx 0.010.$$

3 No answer given.

4 **a)** $x''(t) = 2t + 3x^2 x' - x'$.

b) Quadratic Taylor with 1 step: $x(2) \approx 1$.
Quadratic Taylor with 2 steps: $x(2) \approx 4$.
Quadratic Taylor with 5 steps: $x(2) \approx 28651.2$.

c) Quadratic Taylor with 10 steps: $x(2) \approx 6 \times 10^{122}$.
Quadratic Taylor with 100 or 1000 steps leads to overflow.

5 **a)** No solution given.

b) $x'''(t) = 2 + 6xx'^2 + 3x^2 x'' - x''$.
One time step: $x(2) \approx 3.66667$.
Two time steps: $x(2) \approx 22.4696$.

c) No solution given.

d) 10 time steps: $x(2) \approx 1.5 \times 10^{938}$ (overflow with 64 bit numbers).
100 time steps: overflow.
1000 time steps: overflow.

Section 14.7

1 **a)** $x(1) \approx 2$.

b) $x(1) \approx 2.5$.

c) $x(1) \approx 2.5$.

d) $x(1) \approx 2.70833$.

e) $x(1) \approx 2.71735$.

f) No answer given.

g) No answer given.

2 **a)** Approximation at $t = 2\pi$:
Euler's method with 1 step: $x(2\pi) \approx 11.0015$.
Euler's method with 2 steps: $x(2\pi) \approx 4.71828$.
Euler's method with 5 steps: $x(2\pi) \approx 0.276243$.
Euler's method with 10 steps: $x(2\pi) \approx 2.14625$.

b) Approximation at $t = 2\pi$:
Euler's midpoint method with 1 step: $x(2\pi) \approx 4.71828$.
Euler's midpoint method with 5 steps: $x(2\pi) \approx 3.89923$.

c) No solution given.

3 No answer given.

4 No answer given.

5 No answer given.

Section 14.9

1 **a)** We set $x_1 = y$, $x_2 = y'$, $x_3 = x$, and $x_4 = x'$. This gives the system

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= x_1^2 - x_3 + e^t, \\ x_3' &= x_4, \\ x_4' &= x_1 - x_3^2 - e^t. \end{aligned}$$

b) We set $x_1 = x$, $x_2 = x'$, $x_3 = y$, and $x_4 = y'$. This gives the system

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= 2x_3 - 4t^2 x_1, \\ x_3' &= x_4, \\ x_4' &= -2x_1 - 2t x_2. \end{aligned}$$

c) No answer given.

d) No answer given.

2 No answer given.

3 **a)** With $x_1 = x$ and $x_2 = x'$ we obtain

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= (-3x_1 - t^2 x_2). \end{aligned}$$

b) With $x_1 = x$ and $x_2 = x'$ we obtain

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= (-k_s x_1 - k_d x_2)/m. \end{aligned}$$

c) No answer given.

d) No answer given.

4 Euler with 2 steps:

$$x(2) \approx 7, \quad x'(2) \approx 6.53657, \quad y(2) \approx -1.33333, \quad y'(2) \approx -8.3619.$$

Euler's midpoint method with 2 steps:

$$x(2) \approx 7.06799, \quad x'(2) \approx -1.0262, \quad y(2) \approx -8.32262, \quad y'(2) \approx -15.2461.$$

5 No answer given.

6 No answer given.

7 No answer given.

8 No answer given.