Taylor Approksimasjon

Gitt en funksjon $f(x)$ og et punkt $a \in \mathbb{R}$

Taylor approks. $P_n(x) = T_n f(x)$: P_n er det entydige polynomet av grad $\leq n$ s.a. $P_n(a) = f(a)$, $P_n'(a) = f'(a)$, ..., $P_n^{(n)}(a) = f^{(n)}(a)$.

Har vist at $P_n(x) = T_n f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!}$

$= f(a) + f'(a)(x-a) + \ldots + \frac{f^{(n)}(a)(x-a)^n}{n!}$.

Restleddet Har vist at

$f(x) = T_n f(x) + R_n(f)$, hvor

$$R_n(f) = \frac{1}{n!} \int_a^x f^{(n+1)}(t)(x-t)^n \, dt.$$

$\Rightarrow$

Øvre grense :

$$|R_n(f)| \leq \frac{M}{(n+1)!}(x-a)^{n+1}$$

hvis $x > a$, $M = \max_{a \leq t \leq x} |f^{(n+1)}(t)|$.

Restledd 2. Anta at $f^{(n+1)}$ er kontinuerlig i $[a,x]$, anta igjen at $x > a$. Da finnes det et reellt tall $c \in (a,x)$ slik at

$$R_n f(x) = \frac{f^{(n+1)}(c)\,(x-a)^{n+1}}{(n+1)!}.$$

Merk:

① Dette ligner det neste leddet i Taylor approksimasjonen: $\dfrac{f^{(n+1)}(a)\,(x-a)^{n+1}}{(n+1)!}$

② c avhenger av x

Bevis. La $M = \max_{a \le t \le x} f^{(n+1)}(t)$, $m = \min_{a \le t \le x} f^{(n+1)}(t)$

Fra integralformen av $R_n f$ får vi at

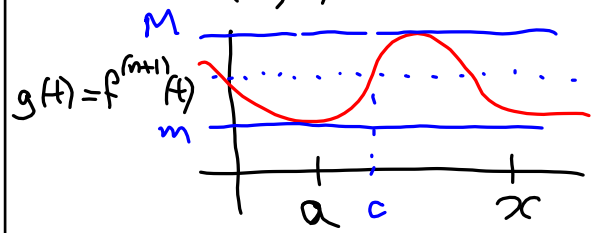
$$R_n(f) \le M \frac{1}{n!}\int_a^x (x-t)^n\,dt = \frac{M(x-a)^{n+1}}{(n+1)!}$$

$$R_n(f) \ge m \frac{1}{n!}\int_a^x (x-t)^n\,dt = \frac{m(x-a)^{n+1}}{(n+1)!}$$

(fordi $(x-t)^n \ge 0$)

$$\Rightarrow \quad m \le \boxed{\frac{(n+1)!\,R_n(f)}{(x-a)^{n+1}}} \le M.$$

Skjæringssetningen sier at p.g.a. at $f^{(n+1)}$ er kontinuerlig må det finnes $c \in (x,a)$ slik at $f^{(n+1)}(c) = \dfrac{(n+1)!\,R_n(f)}{(x-a)^{n+1}}$.



$$\Rightarrow \quad R_n(f) = \frac{f^{(n+1)}(c)\,(x-a)^{n+1}}{(n+1)!} \qquad \square$$

Eksempel. Beregn $I = \int_0^1 \frac{\sin x}{x}\,dx$ med en feil $< 10^{-4}$.

Ingen analytisk løsning. Må løse problemet numerisk. Ide: approksimere $\sin x$.

Velger n og bruker $\sin x \approx T_n \sin x$.

$T_n \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!}$ (og $a=0$)

Da er

$$I \approx \int_0^1 \frac{T_n \sin x}{x}\,dx = \int_0^1 1 - \frac{x^2}{3!} + \dots \frac{(-1)x^{2n}}{(2n+1)!}\,dx$$

$$= \left[x - \frac{x^3}{3\cdot 3!} + \dots \frac{(-1)x^{2n+1}}{(2n+1)(2n+1)!} \right]_{x=0}^{x=1}$$

$$= 1 - \frac{1}{3\cdot 3!} + \dots + \frac{(-1)}{(2n+1)(2n+1)!}$$

Estimerer feilen fra restleddet.

$$R_n(f) = \frac{f^{(2n+1)}(c)\,x^{2n+1}}{(2n+1)!}$$

$$\left| \int_0^1 \frac{\sin x}{x} - \int_0^1 \frac{T_n \sin x}{x} \right| = \left| \int_0^1 \frac{R_n f(x)}{x} \right|$$

$$\le \int_0^1 \left| \frac{R_n f(x)}{x} \right| dx \le \int_0^1 \frac{x^{2n+1}}{(2n+1)!\,x}\,dx$$

(fordi alle deriverte til $\sin x \le 1$ i absoluttverdi)

$$= \int_0^1 \frac{x^{2n}}{(2n+1)!}\,dx = \left[\frac{x^{2n+1}}{(2n+1)(2n+1)!} \right]_0^1$$

$$= \frac{1}{(2n+1)(2n+1)!}$$

Ønsker å velge n slik at dette er mindre enn 10^{-4}.

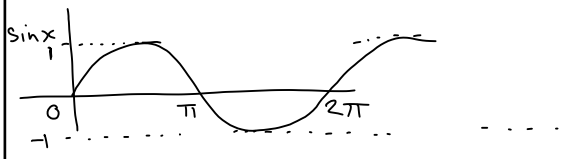
Prøv:

n	$\frac{1}{(2n+1)(2n+1)!}$
1	$\frac{1}{3\cdot 3!} = \frac{1}{18} \approx 5.6 \times 10^{-2}$
2	$\frac{1}{5\cdot 5!} \approx 1.7 \times 10^{-3}$
3	$\frac{1}{7\cdot 7!} \approx 2.8 \times 10^{-5}$

Da er $n=3$ tilstrekkelig.

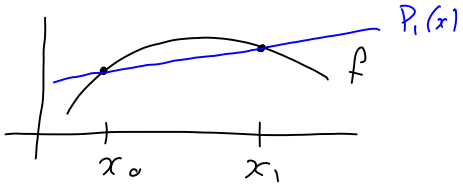
d.v.s. $I \approx 1 - \frac{1}{3\cdot 3!} + \frac{1}{5\cdot 5!} = 0.9461$

med nøyaktighet bedre enn 10^{-4}.

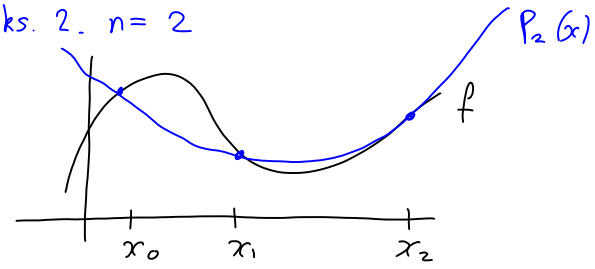


Interpolasjon. Anta at vi har verdiene $f(x_0), f(x_1), \ldots, f(x_n)$ av en funksjon f i punktene $x_0, x_1, \ldots, x_n$. Da finnes det et entydig polynom $P_n(x)$ av grad $\leq n$ slik at $P_n(x_i) = f(x_i)$, alle $i = 0, 1, \ldots, n$.

Eksempel 1. $n = 1$.



Eks. 2. $n = 2$



Hvordan finne $P_n(x)$?

$$P_n(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n$$

Eksempel $n = 2$. $P_2(x) = c_0 + c_1 x + c_2 x^2$.

Anta at $x_0 = 0$, $x_1 = 1$, $x_2 = 2$

Anta at $f(0) = 1$, $f(1) = 3$, $f(2) = 2$.

$1 = f(0) = P_2(0) = c_0 \Rightarrow c_0 = 1$

$3 = f(1) = P_2(1) = c_0 + c_1 + c_2 = 1 + c_1 + c_2$

$\Rightarrow \boxed{c_1 + c_2 = 2}$

$2 = f(2) = P_2(2) = c_0 + 2c_1 + 4c_2 \Rightarrow \boxed{2c_1 + 4c_2 = 1}$

$\Rightarrow c_1 = \frac{7}{2}, \quad c_2 = -\frac{3}{2}.$

$\Rightarrow P_2(x) = 1 + \frac{7}{2}x - \frac{3}{2}x^2.$ (*)

Bedre å bruke Newtonformen av $P_n(x)$.

$P_2(x) = c_0 + c_1 x + c_2 x(x-1).$

$1 = f(0) = P_2(0) = c_0 \Rightarrow c_0 = 1$

$3 = f(1) = P_2(1) = c_0 + c_1 + c_2 \cdot 0 = 1 + c_1 \Rightarrow c_1 = 2$

$2 = f(2) = P_2(2) = c_0 + 2c_1 + 2c_2 = 1 + 4 + 2c_2$

$\Rightarrow 2c_2 = -3, \quad c_2 = -\frac{3}{2}.$

$\Rightarrow P_2(x) = 1 + 2x - \frac{3}{2}x(x-1).$ (*)

$= 1 + \frac{7}{2}x - \frac{3}{2}x^2$

Generelt: Newtonformen er

$P_n(x) = c_0 + c_1(x-x_0) + c_2(x-x_0)(x-x_1)$

$+ \ldots + c_n(x-x_0)(x-x_1)\ldots(x-x_{n-1})$