Forelesning 22/11
Eksamen 2022 forts.

7. $y'' - y' - 6y = 3x - \frac{3}{2}$ $\quad y(0) = \frac{1}{3}$ $\quad y'(0) = 2$

part. løsning: $y_p(x) = Ax + B$, $y_p'(x) = A$, $y_p''(x) = 0$

$$y_p'' - y_p' - 6y_p = -A - 6Ax - 6B = \underbrace{-6Ax}_{3} \underbrace{-A-6B}_{-\frac{3}{2}}$$

$\Downarrow$

$A = -\frac{1}{2}$ $\quad \frac{1}{2} - 6B = -\frac{3}{2}$

$6B = 2$

$B = \frac{1}{3}$

$\Rightarrow y_p(x) = -\frac{1}{2}x + \frac{1}{3}$ (kan egentlig stoppe her)

homogen løsning: $r^2 - r - 6 = 0$

$r = \frac{1 \pm \sqrt{1+24}}{2} = \frac{1 \pm 5}{2} \Rightarrow r_1 = 3,\ r_2 = -2$

Generell løsning: $y(x) = y_p(x) + y_h(x)$

$$= -\frac{1}{2}x + \frac{1}{3} + Ce^{3x} + De^{-2x} \quad \Big| \ \frac{1}{3} + C + D = \frac{1}{3}$$

$$y'(x) = -\frac{1}{2} + 3Ce^{3x} - 2De^{-2x} \quad \Big| \ -\frac{1}{2} + 3C - 2D = 2$$

$C + D = 0$

$3C - 2D = \frac{5}{2}$

$3C + 2C = \frac{5}{2}$

$C = \frac{1}{2},\ D = -\frac{1}{2}$

$\Rightarrow y(x) = -\frac{1}{2}x + 3 + \frac{1}{2}(e^{3x} - e^{-2x})$

(A)

8. $x'' + (t + x^2)x' - e^{2t}x = \sin t$

$x' = y$

$y' + (t + x^2)y - e^{2t}x = \sin t$

$$\Big\} \quad \begin{aligned} x' &= y \\ y' &= -(t+x^2)y + e^{2t}x + \sin t \end{aligned}$$

(B)

9. $(1+t)x' = xe^t$, $x(0)=1$ $\quad\left(x' = \underbrace{\frac{xe^t}{1+t}}_{f(t,x)},\ x(0)=1,\ t_0=0,\ t_{1/2}=\frac{1}{2},\ t_1=1\right)$

Ett steg Euler midtpunkt:

$$x_{1/2} = x_0 + \underbrace{\tfrac{1}{2}}_{h/2} f(t_0,x_0) = 1 + \tfrac{1}{2}\frac{1\cdot e^0}{1+0} = 1+\tfrac{1}{2} = \tfrac{3}{2}$$

$$x_1 = x_0 + 1\cdot f(t_{1/2},x_{1/2}) = 1 + \frac{\frac{3}{2}e^{1/2}}{1+\frac{1}{2}} = 1 + \frac{\frac{3}{2}e^{\frac{1}{2}}}{\frac{3}{2}} = 1+e^{\frac{1}{2}}$$

(C)

10. $\max_{x\in[-1,1]} |f^{(n)}(x)| \le n^2$, $n \ge 1$. Vi skal Taylorutvikler fom 0.

$$R_n(x) = \frac{|f^{(n+1)}(c)|}{(n+1)!}x^{n+1}$$

for $x\in[-1,1]$: $|R_n(x)| = \frac{|f^{(n+1)}(c)|}{(n+1)!}x^{n+1} \le \frac{(n+1)^2}{(n+1)!}|x|^{n+1} = \frac{n+1}{n!}|x|^{n+1} \le \frac{n+1}{n!}$

$\Downarrow$ $c\in[-1,1]$

Vi vil at $|R_n f(x)| < 10^{-3}$

Dette får vi til hvis $\quad \frac{n+1}{n!} < 10^{-3}$

$\Updownarrow$

$$n! > 1000(n+1)$$

$n=0$	1	
$n=1$	1	
$n=2$	2	
⋮	⋮	
$n=6$	720	
$n=7$	5040	$1000\cdot 8 = 8000$
$n=8$	40320	$1000\cdot 9 = 9000$

minste n som garantert går: $n=8$ (D)

Del 2

1. Vis ved induksjon at $\underbrace{2^{n+1} > n^2}_{P_n}$, $n \ge 3$

P_3 sier at $2^{3+1} > 3^2 \Leftrightarrow 16 > 9$, slik at P_3 er sann.

Anta at vi har vist at $P_3, \ldots, P_k$ er sanne.
Vi må vise at P_{k+1} også er sann, det vil si at

$$2^{(k+1)+1} > (k+1)^2$$

Siden vi antar P_k sann, så har vi at $2^{k+1} > k^2$ | gang med $\frac{(k+1)^2}{k^2}$, begge sider.

$$2^{(k+1)+1} = 2 \cdot 2^{k+1} = (\sqrt{2})^2 2^{k+1} > \underbrace{\left(1+\frac{1}{k}\right)^2}_{<\sqrt{2} \text{ for } k \geq 3} 2^{k+1} = \frac{(k+1)^2}{k^2} 2^{k+1} > (k+1)^2$$

Dette viser at P_{k+1} er sann. Dermed er P_n sann for alle $n \geq 3$.

2. $8x_{n+2} - 33x_{n+1} + 4x_n = 21n - 4 \qquad x_0 = 9,\ x_1 = 1$

kar. likning: $8r^2 - 33r + 4 = 0 \Rightarrow r = \frac{33 \pm \sqrt{1089 - 128}}{16} = \frac{33 \pm 31}{16}$

$\Rightarrow r_1 = 4,\ r_2 = \frac{1}{8}$

$x_n^h = C4^n + D8^{-n}$

part.løsning: $x_n^p = An + B$:

$$8A(n+2) + 8B - 33A(n+1) - 33B + 4An + 4B$$
$$= (8A - 33A + 4A)n + 16A + 8B - 33A - 33B + 4B$$
$$= \underbrace{-21An}_{21} \underbrace{-17A - 21B}_{-4}$$

$\Downarrow$

$A = -1 \qquad 17 - 21B = -4 \Rightarrow -21B = -21 \Rightarrow B = 1$

$x_n^p = -n + 1.$

Generell løsning: $x_n = x_n^h + x_n^p = -n + 1 + C4^n + D8^{-n}$

$$\left.\begin{matrix} x_0 = 9 \\ x_1 = 1 \end{matrix}\right\} \quad \left.\begin{matrix} 1 + C + D = 9 \\ 4C + \frac{D}{8} = 1 \end{matrix}\right\} \quad \left.\begin{matrix} C + D = 8 \\ 32C + D = 8 \end{matrix}\right\} \quad C = 0,\ D = 8$$

$$\Rightarrow x_n = -n + 1 + 8 \cdot 8^{-n} = -n + 1 + 8^{1-n}$$

b) Maskinen regner ut $x_{n+2} = (33x_{n+1} - 4x_n + 21n - 4)/8$

Ingen avrundingsfeil i koeffisienter og start betingelser
Maskinen vil derfor følge løsningsformelen eksakt til å begynne med.
Til slutt: vi får avrundingsfeil likevel
($-n+1 - 8^{1-n}$ vil bruke opp de tilgjengelige 53 signifikante bits).
Fra da av vil også den andre roten bidra (med $\varepsilon 4^n$)

3. $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0 \quad \Rightarrow f'(0) = a_1$

$$\begin{aligned} f(0) &= a_0 && \cdot(-3)\\ f(h) &= a_3h^3 + a_2h^2 + a_1h + a_0 && \cdot 4\\ f(2h) &= 8a_3h^3 + 4a_2h^2 + 2a_1h + a_0 && \cdot(-1)\\ & -4a_3h^3 \quad 0 \quad 2a_1h \quad 0 \end{aligned}$$

$$f'(0) - \frac{-3f(0) + 4f(h) - f(2h)}{2h} = a_1 - \frac{-4a_3h^3 + 2a_1h}{2h} = a_1 + 2a_3h^2 - a_1$$
$$= 2a_3h^2$$
blir 0

Når f er andregradspolynom så er $a_3 = 0$, slik at

$$f'(0) = \frac{-3f(0) + 4f(h) - 2f(h)}{2h}$$

4. $f(x) = 3 + 2x - x^2 + x^3 \qquad f(-1) = -1,\ f(0) = 3,\ f(2) = 11$

finn $p_2(x)$ (grad ≤ 2 s.a. $p_2(-1) = -1,\ p_2(0) = 3,\ p_2(2) = 11$

$p_2(x) = c_0 + c_1(x+1) + c_2(x+1)x$

$$\begin{aligned} -1 &= c_0\\ 3 &= c_0 + c_1 && \Rightarrow c_1 = 3 - c_0 = 4\\ 11 &= c_0 + 3c_1 + 6c_2 && \Rightarrow c_2 = (11 - c_0 - 3c_1)/6\\ & && = (11 + 1 - 12)/6 = 0 \end{aligned}$$

$p_2(x) = -1 + 4(x+1)$

5. $x' = \frac{x}{1+4t^2} \qquad t \ge 0 \qquad x(0) = 1$

a. Løsning: Likningen er separabel:

$$\frac{x'}{x} = \frac{1}{1+4t^2}$$

$u = 2t,\ du = 2dt$

$$\frac{1}{2}\int \frac{1}{1+u^2}\,du = \frac{1}{2}\arctan u + C = \frac{1}{2}\arctan(2t) + C$$

$$\ln|x| = \frac{1}{2}\arctan(2t) + C$$

$$|x| = e^{\frac{1}{2}\arctan(2t)+C} = k\,e^{\frac{1}{2}\arctan 2t}$$

$$x(t) = k' e^{\frac{1}{2}\arctan 2t}$$

$$x(0)=1 \Rightarrow k' = 1$$

$$x(t) = e^{\frac{1}{2}\arctan 2t}$$

b. $h = \frac{1}{4}$, Euler to steg:

$$x_1 = x_0 + \frac{1}{4} f(t_0, x_0) = 1 + \frac{1}{4}\cdot\frac{1}{1+4\cdot 0^2} = 1 + \frac{1}{4} = \frac{5}{4}$$

$$x_2 = x_1 + \frac{1}{4} f(t_1, x_1) = \frac{5}{4} + \frac{1}{4}\cdot\frac{\frac{5}{4}}{1+4\cdot(\frac{1}{4})^2} = \frac{5}{4} + \frac{1}{4}\cdot\frac{20}{16+4}$$

$$= \frac{5}{4} + \frac{1}{4} = \frac{3}{2}$$

Fel: $|x_2 - x(\frac{1}{2})| = |\frac{3}{2} - 1.48097| = 0.0190$