Compulsory Project 1
MAT-INF3100 Linear optimization, Spring 2016

- **NOTE.** Send the project by *Thursday February 18 at 15:00* to Torkel Haufmann (torkelah@math.uio.no) in a single PDF file that you name "username.pdf" (your username!). Also attach your computer code, either as a listing in the PDF file or (preferrably) as separate files. Moreover, you should read the general information about compulsory projects at the course web page.

Problem 1

1a)

Consider the LP problem

$$\begin{array}{lrcrcrcl} \text{maximize} & -7x_1 & & & + & 2x_3 & & \\ \text{subject to} & & & & & & & \\ & & - & 3x_2 & + & 4x_3 & \leq & 1, \\ & x_1 & - & x_2 & & & \leq & 2, \\ & -3x_1 & & & + & x_3 & \leq & 0, \\ & & & & & x_1, x_2, x_3 & \geq & 0. \end{array} \tag{1}$$

Identify vectors x, c, b and a matrix A such that (1) can be written

$$\max c^T x \quad \text{subject to} \quad Ax \leq b,\ x \geq 0.$$

1b)

Use the simplex algorithm to find an optimal solution.

1c)

Consider the LP problem

$$\begin{aligned} &\max \sum_{j=1}^{n} c_j x_j, \\ &\text{subject to} \\ &\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m, \\ &x_j \geq 0, \quad j = 1, \ldots, n, \end{aligned} \tag{2}$$

where c_j, a_{ij}, b_i are given numbers. Introduce slack variables $x_{n+1}, \ldots, x_{n+m}$ and identify vectors x, c, b and a matrix A such that (2) can be written

$$\max c^T x \quad \text{subject to} \quad Ax = b,\ x \geq 0. \tag{3}$$

Problem 2

Consider the LP problem

$$
\begin{array}{lrcrcl}
\text{maximize} & -3x_1 & + & 6x_2 & & \\
\text{subject to} & & & & & \\
 & 2x_1 & + & x_2 & \leq & 6, \\
 & -x_1 & + & 2x_2 & \leq & 2, \\
 & & & x_1, x_2 & \geq & 0.
\end{array}
\tag{4}
$$

2a)

Use the simplex algorithm to find all optimal solutions.

2b)

Draw the feasible region of the LP problem (4) and highlight the optimal solutions in your drawing. Explain why the non-uniqueness of optimal solutions occurs.

2c)

Use the simplex algorithm to show that the following LP problem is unbounded:

$$
\begin{array}{lrcrcl}
\text{maximize} & 3x_1 & + & 2x_2 & & \\
\text{subject to} & & & & & \\
 & x_1 & - & x_2 & \leq & 3, \\
 & x_1 & & & \leq & 2, \\
 & & & x_1, x_2 & \geq & 0.
\end{array}
\tag{5}
$$

Problem 3 - Linear Regression

Let $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$ be some given points in the plane, where b_i is assumed to (somehow) be a function of a_i. In this exercise we will study the *linear regression* approach to finding a function of a given kind best matching the observed points. For more details, see chapter 12 in the book.

We want to find a line $y = x_1 r + x_2$ which is a best match to the observations (Here x_1 and x_2 are the coefficients of the line, and r takes the values $a_1, a_2, \ldots, a_n$). For technical reasons, we instead model a line as a function of two variables, $y = x_1 r_1 + x_2 r_2$, and assume $r_2 = 1$ at every point.

This means we want to determine values of x_1 and x_2 giving the "best possible" solution to the overdetermined set of equalities

$$
\begin{aligned}
a_1 x_1 + 1x_2 &= b_1, \\
a_2 x_1 + 1x_2 &= b_2, \\
&\vdots \\
a_n x_1 + 1x_2 &= b_n.
\end{aligned}
$$

We define

$$A = \begin{bmatrix} a_1 & 1 \\ a_2 & 1 \\ \vdots & \vdots \\ a_n & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

If $\|\cdot\|$ is some norm on $\mathbb{R}^n$ we are looking to find the vector $x \in \mathbb{R}^2$ minimizing $\|b - Ax\|$.

Recall the definitions of the 1-norm and 2-norm:

$$\|z\|_1 = \sum_i |z_i|, \qquad \|z\|_2 = \sqrt{\sum_i z_i^2}.$$

L_1-regression is regression using the 1-norm, and similarly L_2-regression uses the 2-norm.

L_2-regression is often called the *method of least squares*, and it can be shown that the optimal solution to

$$\arg\min_x \|b - Ax\|_2^2 \tag{6}$$

is given by $(A^T A)^{-1} A^T b$ (in (6) we square the norm, but that does not affect the optimal solution).

For the L_1-regression there is no such simple formula for the optimal solution, but we can solve a linear program: Writing it out we see that the L_1-regression problem is

$$\text{minimize} \sum_i |b_i - \sum_j a_{ij} x_j|. \tag{7}$$

3a)

Explain why (7) can be rewritten as

$$\begin{aligned} &\text{minimize} \sum_i t_i \\ &\text{subject to } -t_i \le b_i - \sum_j a_{ij} x_j \le t_i, \quad i = 1, 2, \ldots, n. \end{aligned} \tag{8}$$

3b)

We will investigate the difference between L_1- and L_2-regression using the following 10 data points (i.e., $n = 10$):

$$(0,1), (1,1), (2,2), (3,4), (4,3), (5,5), (6,6), (7,10), (8,8), (9,10).$$

- Solve (6) by using the formula $(A^T A)^{-1} A^T b$.
- Solve (8) using OPL-CPLEX. *(Note: in* (8) *the variables are not assumed to be nonnegative.)*
- Plot the two lines along with the data points.
- Compare the two lines. Is the difference as expected? Is there a reason to prefer one method over the other?