

UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT-INF3600 — Mathematical logic.

Day of examination: Monday, December 19, 2022.

Examination hours: 15:00 – 19:00.

This problem set consists of 3 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

The weights might be adjusted.

PART I

Let Q be a unary relation symbol, let S be a unary function symbol, let 0 be a constant symbol and let $\mathcal{L}$ be the language $\{0, S, Q\}$. Furthermore, let

$$\Sigma_1 = \{ \forall x [Q(Sx) \rightarrow Q(x)], Q(S0) \}.$$

Problem 1 (weight 10 %)

Give a full Σ_1 -deduction of $Q(0)$. Name all the logical axioms and inference rules involved in the deduction.

Problem 2 (weight 10 %)

Give a full Σ_1 -deduction of $0 = x \rightarrow Q(x)$. Name all the logical axioms and inference rules involved in the deduction.

For any natural number n , we define the numeral $\bar{n}$ by $\bar{0} = 0$ and $\overline{n+1} = S\bar{n}$. Let

$$\Sigma_n = \{ \forall x [Q(Sx) \rightarrow Q(x)], Q(\bar{n}) \}.$$

Problem 3 (weight 10 %)

Do we have $\Sigma_{17} \models \Sigma_{16}$? Do we have $\Sigma_{16} \models \Sigma_{17}$? Justify your answers.

Let $\Sigma = \bigcup_{i \in \mathbb{N}} \Sigma_i$.

(Continued on page 2.)

Problem 4 (weight 10 %)

Give an $\mathcal{L}$ -sentence ϕ such that $\Sigma \not\models \phi$ and $\Sigma \not\models \neg\phi$. Prove that we indeed have $\Sigma \not\models \phi$ and $\Sigma \not\models \neg\phi$.

PART II

Let f be a binary function symbol, let c be a constant symbol and let $\mathcal{L}$ be the language $\{c, f\}$. Let

- $\Gamma_1 = \{ \forall x [x = c], f(c, c) = c \}$
- $\Gamma_2 = \{ \forall x [x = c], \neg(f(c, c) = c) \}$
- $\Gamma_3 = \{ \forall x [\neg(x = c)], \neg(f(c, c) = c) \}$
- $\Gamma_4 = \{ \neg\forall x [x = c], f(c, c) = c \}$.

Problem 5 (weight 10 %)

What does it mean that a set of first-order formulas is consistent? (Give the definition.) Is Γ_1 consistent? Is Γ_2 consistent? Is Γ_3 consistent? Is Γ_4 consistent? Justify your answers.

For any $\mathcal{L}$ -term t , let $(t)_y^x$ denote the term t where every occurrence of the variable x is replaced by the variable y . E.g., $(ffv_1cfv_2v_1)_{v_3}^{v_1}$ denotes the term $ffv_3cfv_2v_3$ and $(fcc)_{v_7}^{v_3}$ denotes the term fcc .

Theorem 1. For any variables x, y and any $\mathcal{L}$ -term t , we have

$$\vdash x = y \rightarrow t = (t)_y^x.$$

Problem 6 (weight 10 %)

Prove Theorem 1. Use induction on the structure of the term t .

Problem 7 (weight 10 %)

Do we have $\vdash t = (t)_y^x \rightarrow x = y$ (for any variables x, y and any $\mathcal{L}$ -term t)? Do we have $\vdash \neg(t = (t)_y^x \rightarrow x = y)$ (for any variables x, y and any $\mathcal{L}$ -term t)? Justify your answers.

PART III

Recall the language $\mathcal{L}_{NT}$, that is, the language $\{0, S, +, \cdot, E, <\}$, and its standard structure $\mathfrak{N}$. Some $\mathcal{L}_{NT}$ -formulas contain bounded quantifiers, and some $\mathcal{L}_{NT}$ -formulas are Δ -formulas.

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Problem 8 (weight 5 %)

What is a bounded quantifier? What is a Δ -formula?

Let $\phi(x) \equiv (\exists y)[y + y = x]$, and let $\psi(x) \equiv \neg(\exists y)[y + y = x]$.

Problem 9 (weight 5 %)

Give a Δ -formula $\phi_0(x)$ such that $\mathfrak{N} \models \phi_0(\bar{a})$ if and only if $\mathfrak{N} \models \phi(\bar{a})$ (for any natural number a). Give a Δ -formula $\psi_0(x)$ such that $\mathfrak{N} \models \psi_0(\bar{a})$ if and only if $\mathfrak{N} \models \psi(\bar{a})$ (for any natural number a).

Let p_i denote the i 'th prime, that is, $p_1 = 2$ and $p_2 = 3$ and so on. We encode a nonempty finite sequence of natural numbers $a_1, a_2, \dots, a_k$ as the single natural number $\langle a_1, a_2, \dots, a_k \rangle$ where

$$\langle a_1, a_2, \dots, a_k \rangle = p_1^{a_1+1} \cdot p_2^{a_2+1} \cdot \dots \cdot p_k^{a_k+1}.$$

The Δ -formula $IthElement(x_1, x_2, x_3)$ is known from Leary & Kristiansen's textbook. We have $\mathfrak{N} \models IthElement(\bar{b}, \bar{i}, \bar{a})$ if and only if there exists a sequence of natural numbers $a_1, a_2, \dots, a_k$ such that $a = \langle a_1, a_2, \dots, a_k \rangle$ and $a_i = b$.

We define the sequence $F_0, F_1, F_2, \dots$ of Fibonacci numbers by $F_0 = F_1 = 1$ and $F_{n+2} = F_n + F_{n+1}$.

Problem 10 (weight 10 %)

Give an $\mathcal{L}_{NT}$ -formula $\theta(x_1, x_2)$ such that $\mathfrak{N} \models \theta(\bar{i}, \bar{m})$ if and only if $F_i = m$. Use the formula $IthElement(x_1, x_2, x_3)$ to construct $\theta(x_1, x_2)$.

Recall the first-order theory N from Leary & Kristiansen's book (N is given by 11 nonlogical $\mathcal{L}_{NT}$ -axioms).

Problem 11 (weight 10 %)

Give an $\mathcal{L}_{NT}$ -formula $\eta(x_1, x_2)$ such that

- (1) $N \vdash \eta(\bar{i}, \bar{m})$ if $F_i = m$
- (2) $N \vdash \neg\eta(\bar{i}, \bar{m})$ if $F_i \neq m$

and explain why (1) and (2) hold.

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