1st October, 2025

MAT-INF3600

Mandatory assignment 1 of 1

Submission deadline

24 th Oct 2025, 14:30 in Canvas (canvas.uio.no).

Instructions

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. Students who fail the assignment, but have made a genuine effort at solving the exercises, are given a second attempt at revising their answers. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

Problem 1. Let c be a constant symbol, let f be unary function symbol, and let R be a binary relation symbol. Let $\mathcal{L}$ be the first-order language $\{c, f, R\}$.

(a) Give an $\mathcal{L}$-structure $\mathfrak{A}$ such that (i) the universe A contains exactly 2 elements and (ii) $\mathfrak{A} \models (\forall x)[R(x, f(x))]$.

(b) Give an $\mathcal{L}$-structure $\mathfrak{A}$ such that (i) the universe A contains exactly 2 elements and (ii) $\mathfrak{A} \not\models (\forall x)[R(x, f(x))]$.

(c) Give an $\mathcal{L}$-structure $\mathfrak{A}$ such that (i) the universe A contains exactly 5 elements and (ii) $\mathfrak{A} \models \forall x \forall y[f(x) = f(y) \to x = y]$.

(d) Let

$$\Gamma \;=\; \{\; \forall x[f(x) \neq c]\,,\; \forall x \forall y[f(x) = f(y) \to x = y]\;\}\;.$$

Give an $\mathcal{L}$-structure $\mathfrak{A}$ such that $\mathfrak{A} \models \Gamma$. Moreover, prove that any model for Γ has an infinite universe.

Problem 2. The following theorem is a generalized version of the Deduction Theorem (Theorem 2.7.4, page 64 in Leary & Kristiansen):

> **Theorem.** Let $n \geq 1$, and let $\theta_1, \ldots, \theta_n$ be sentences. Furthermore, let Σ be a set of formulas. Then, we have $\Sigma \cup \{\theta_1, \ldots, \theta_n\} \vdash \phi$ if and only if $\Sigma \vdash \theta_1 \wedge \ldots \wedge \theta_n \to \phi$.

Prove the theorem. Use the Deduction Theorem as a lemma in your proof (do not prove the Deduction Theorem).

Problem 3. Let R and S be unary relation symbols.

(a) Prove that

$$\forall x[Rx \to Sx] \vdash \forall y[\neg Sy \to \neg Ry]\;.$$

Give a detailed $\{\forall x[Rx \to Sx]\}$-deduction of $\forall y[\neg Sy \to \neg Ry]$. Name all the logical axioms, and all the rules of inference, involved in the deduction.

(b) Prove that

$$\forall x[Rx \to Sx] \nvdash \forall y[Sy \to Ry] \ .$$

Problem 4. Let $\bot$ denote the formula $(\forall x)x = x \wedge \neg(\forall x)x = x$. The following lemmas are known from our textbook:

- **Lemma 1.** $\Sigma \vdash \theta$ if and only if $\Sigma \cup \{\neg\theta\} \vdash \bot$.
- **Lemma 2.** $\Sigma \vdash \theta$ if and only if $\Sigma \vdash \forall x\theta$.

You may refer to these two lemmas when you are answering problem **a** and **b** (do not prove the lemmas). Try to avoid to refer to other lemmas in our textbook.

Let R and S be unary relation symbols (and recall that $(\exists x)(\phi) :\equiv (\neg(\forall x)((\neg\phi)))$).

(a) Prove that $\vdash (\exists x)[\ R(x)\ ] \to (\exists x)[\ R(x) \vee S(x)\ ]$.

(b) Prove that $\vdash (\exists x)[\ R(x) \to (\forall x)[\ R(x)\ ]\ ]$.

Problem 5. Let 0 be a constant symbol, let v and h be unary function symbols, and let f be a binary function symbol. Furthermore, let $\mathcal{L}$ be the first-order language $\{0, v, h, f\}$, and let T be the $\mathcal{L}$-theory consisting of the following non-logical axioms:

(T1) $v(0) = 0$

(T2) $h(0) = 0$

(T3) $(\forall x)(\forall y)[\ v(f(x, y)) = x\ ]$

(T4) $(\forall x)(\forall y)[\ h(f(x, y)) = y\ ]$.

Next, we give an inductive definition of the *prime terms*:

- 0 is a prime term
- $f(t_1, t_2)$ is a prime term if t_1 and t_2 are prime terms.

Theorem. For each variable-free $\mathcal{L}$-term t, there exists a prime term p such that $T \vdash t = p$.

Prove the theorem.