UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Exam in:	MAT-INF3600 — Mathematical logic.
Day of exam:	Monday, December 8, 2008.
Exam hours:	9.00 – 12.00.
This examination set consists of 2 pages.	
Appendices:	None.
Permitted aids:	None.

Make sure that your copy of this examination set is complete before answering.

Problem 1

a) A language is a set of strings over an alphabet. What does it mean that a language is recursive? What does it mean that a language is recursively enumerable? Give the essence of the definitions from Lewis & Papadimitriou's textbook. Give short answers.

b) What does it mean that a function $f : \mathbb{N} \times \ldots \times \mathbb{N} \to \mathbb{N}$ is primitive recursive? Give a short answer.

c) Let $f(x, y) = x + y$ and $g(x, y) = x \times y$. Prove that f and g are primitive recursive functions.

Problem 2

Let S be a unary function symbol, and let a and 0 be constant symbols. Let $\mathcal{L}$ be the first-order language $\{a, 0, S\}$, and let Σ_0 be the $\mathcal{L}$-theory consisting of the following three non-logical axioms:

- $\forall x[0 \neq Sx]$
- $\forall xy[Sx = Sy \;\rightarrow\; x = y]$
- $a = Sa$.

(Continued on page 2.)

a) Give two $\mathcal{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathfrak{A} \models \Sigma_0$ and $\mathfrak{B} \not\models \Sigma_0$.

Lemma 1. Let $t_1, \ldots, t_n$ be substitutable for respectively $x_1, \ldots, x_n$ in the $\mathcal{L}$-formula ϕ. Let $\phi^{x_1,\ldots,x_n}_{t_1,\ldots,t_n}$ denote ϕ where $x_1, \ldots, x_n$ are replaced by respectively $t_1, \ldots, t_n$. Then, $\forall x_1, \ldots, x_n \phi \vdash \phi^{x_1,\ldots,x_n}_{t_1,\ldots,t_n}$.

b) Prove Lemma 1.

c) Prove that $\Sigma_0 \vdash 0 \neq a$ by providing a derivation. The proof should not refer to any lemmas or theorems from the textbook, but you can refer to Lemma 1.

d) Let $\overline{0} = 0$ and $\overline{n+1} = S\overline{n}$.

Prove that $\Sigma_0 \vdash a \neq \overline{n}$ for any $n \in \mathbb{N}$. Use induction on n.

We extend $\mathcal{L}$ with a unary function symbol f and a binary relation symbol $\sqsubseteq$. Let Σ_1 be Σ_0 extended with the non-logical axioms

- $x \sqsubseteq y \ \leftrightarrow \ x = a \vee x = y$
- $x \sqsubseteq y \ \rightarrow \ f(x) \sqsubseteq f(y)$.

Lemma 2. For any $\mathcal{L}$-formula ϕ, any $\mathcal{L}$-sentence η and any variable x, we have

1. $\vdash \phi$ if and only if $\vdash \forall x \phi$
2. $\eta \vdash \phi$ if and only if $\vdash \eta \rightarrow \phi$.

e) Prove that

$$\Sigma_1 \vdash f(a) \neq a \ \rightarrow \ \forall x[f(a) = f(x)]$$

by providing a derivation. You might find Lemma 1 and Lemma 2 helpful.

f) Prove that there exists a closed quantifier-free formula ξ such that $\Sigma_1 \nvdash \xi$ and $\Sigma_1 \nvdash \neg\xi$. (A formula is closed when there are no free variables in the formula.)

g) Let Σ_2 be Σ_1 extended with the non-logical axiom $f(a) = 0$.

Prove the following assertion: For any closed $\mathcal{L}$-term t there exists $n \in \mathbb{N}$ such that $\Sigma_2 \vdash t = \overline{n} \ \vee \ t = a$.

h) Let

$$\Gamma = \{\phi \mid \phi \text{ is a closed quantifier-free formula and } \Sigma_2 \vdash \phi\} \, .$$

The set Γ can obviously be regarded as a language, that is, as a set of strings over an alphabet. Prove that Γ is a recursive language.

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