UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT-INF3600 — Mathematical logic.

Day of examination: Thursday, December 9, 2010.

Examination hours: 9.00 – 13.00.

This problem set consists of 3 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

The following theorem is known from Leary's book.

> **Theorem (A).** Let $\mathcal{L}$ be a first-order language. For any $\mathcal{L}$-theory T, any variable x, and any $\mathcal{L}$-formula ϕ, we have
>
> $$T \vdash \forall x \phi \quad \Leftrightarrow \quad T \vdash \phi \, .$$

Problem 1

Prove Theorem (A).

Let 0 be a constant symbol, let $'$ be a unary function symbol, and let $<$ be a binary relation symbol. Let $\mathcal{L}$ be the first-order language $\{0, ', <\}$, and let T be the $\mathcal{L}$-theory consisting of the following non-logical axioms:

(T1) $\forall x \, [\neg x < x]$

(T2) $\forall xyz \, [x < y \, \wedge \, y < z \, \rightarrow x < z]$

(T3) $\forall xy \, [x < y \, \rightarrow x' < y']$

(T4) $0 < 0'$

Problem 2

Give a T-derivation of $0' < 0''$. Name the logical and the non-logical axioms involved in the derivation.

(Continued on page 2.)

Problem 3

Give a T-derivation of $\forall xy[x < y \rightarrow \neg x = y]$. Name the logical and the non-logical axioms involved in the derivation. You may apply Theorem (A). [Hint: You will need the logical axiom $x = x \wedge x = y \;\rightarrow\; (x < y \;\rightarrow\; x < x)$ (E3).]

For $n \in \mathbb{N}$, we define the $\mathcal{L}$-term $0^{[n]}$ by $0^{[0]} \equiv 0$ and $0^{[m+1]} \equiv 0^{[m]\prime}$.

Problem 4

Prove that

$$m < n \;\Rightarrow\; T \vdash \neg\, 0^{[m]} = 0^{[n]} \,.$$

[Hint: Use induction on n to prove that we have $T \vdash 0^{[i]} < 0^{[n+1]}$ for any $i \leq n$.]

Let T^- denote T without the axiom (T4), that is, T^- is the first-order theory consisting of the axioms (T1), (T2) and (T3).

Problem 5

Give a model $\mathfrak{A}$ for T^- such that the universe of $\mathfrak{A}$ contains exactly 17 elements.

Problem 6

Explain why (T4) is independent of the other axioms in T, that is, explain why $T^- \not\vdash (T4)$, and explain why $T^- \not\vdash \neg(T4)$.

We define the $\mathcal{L}$-structure $\mathfrak{N}$: The universe N is the set of natural numbers, that is, $\{0, 1, 2, \ldots\}$. The relation $<^{\mathfrak{N}}$ is the standard strict ordering of the natural numbers. The function ${\prime}^{\mathfrak{N}}$ is the successor function, that is, $n^{\prime^{\mathfrak{N}}} = n + 1$ for any $n \in N$. Finally, $0^{\mathfrak{N}} = 0$.

Recall that an $\mathcal{L}$-structure $\mathfrak{A}$ is *elementary equivalent* to an $\mathcal{L}$-structure $\mathfrak{B}$ if, and only if, we have

$$\mathfrak{A} \models \phi \quad \Leftrightarrow \quad \mathfrak{B} \models \phi$$

for any $\mathcal{L}$-sentence ϕ.

(Continued on page 3.)

Problem 7

Give a model $\mathfrak{A}$ for T that is not elementary equivalent to $\mathfrak{N}$. Give an $\mathcal{L}$-sentence ϕ such that $\mathfrak{A} \models \phi$ and $\mathfrak{N} \models \neg\phi$.

Problem 8

Construct a countable model $\mathfrak{A}$ for T that is elementary equivalent, but not isomorphic, to $\mathfrak{N}$. [Hint: Use the Compactness Theorem.]

Problem 9

Give a standard definition of the primitive recursive functions. You can e.g. give the definition from Lewis's & Padimitriou's book.

We define the bounded quantifiers in the standard way, that is $\forall x < y[\phi] \equiv \forall x[x < y \to \phi]$ and $\exists x < y[\phi] \equiv \exists x[x < y \wedge \phi]$. A Δ_0-formula is an $\mathcal{L}$-formula where all quantifiers are bounded.

> **Theorem (B).** For any Δ_0-formula $\phi(x_1, \ldots, x_n)$, there exists a primitive recursive function f_ϕ such that
>
> - $\mathfrak{N} \models \phi(0^{[a_1]}, \ldots, 0^{[a_n]}) \quad \Leftrightarrow \quad f_\phi(a_1, \ldots, a_n) = 0$
> - $\mathfrak{N} \models \neg\phi(0^{[a_1]}, \ldots, 0^{[a_n]}) \quad \Leftrightarrow \quad f_\phi(a_1, \ldots, a_n) = 1.$

Problem 10

Prove Theorem (B). You can refer to well-known facts about the primitive recursive functions, e.g., to facts stated in Lewis's & Padimitriou's book.

Problem 11

Do you think Theorem (B) holds for $\mathcal{L}$-formulas in general, that is, is the theorem still true if we replace "any Δ_0-formula $\phi(x_1, \ldots, x_n)$" by "any $\mathcal{L}$-formula $\phi(x_1, \ldots, x_n)$"? Answer short, but justify your answer.

END