UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in MAT-INF3600 — Mathematical logic.

Day of examination: Thursday, December 8, 2011.

Examination hours: 9:00 – 13:00.

This problem set consists of 2 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Part I

Theorem. Let Σ be a set of $\mathcal{L}$-formulas, and let ϕ be an $\mathcal{L}$-sentence. Then,

(i) $\Sigma \vdash \phi$ if, and only if, $\Sigma \models \phi$

(ii) Σ has a model if, and only if, Σ is consistent

(iii) if every finite subset of Σ has a model, then Σ has a model.

Problem 1

Explain briefly what it means that a set of formulas is consistent.

Problem 2

Prove that (i) and (ii) are equivalent.

Problem 3

Prove that (i) implies (iii).

Part II

Lemma (A). Let $\mathcal{L}$ be a first-order language. For any $\mathcal{L}$-theory T, any $\mathcal{L}$-terms s, t, u, we have

(i) $T \vdash s = s$

(ii) $T \vdash s = t \land t = u \ \rightarrow \ s = u$

(iii) $T \vdash s = t \ \rightarrow \ t = s$.

(Continued on page 2.)

Problem 1

Prove Clause (i) of Lemma (A) by giving a T-derivation where s is an arbitrary $\mathcal{L}$-term.

Let $\mathcal{L}$ be the first-order language $\{\leq\}$ where $\leq$ is a binary relation symbol, and let T be the $\mathcal{L}$-theory consisting of the non-logical axioms

(A_1) $\forall x[x \leq x]$

(A_2) $\forall xyz[x \leq y \ \wedge \ y \leq z \ \rightarrow \ x \leq z]$

(A_3) $\forall xy[x \leq y \ \wedge \ y \leq x \ \rightarrow \ x = y]$.

Lemma (B). $T \vdash \forall xy[x \leq y \ \wedge \ y \leq x \ \leftrightarrow \ x = y]$.

Problem 2

Prove Lemma (B) by giving a T-derivation. [Hint: use Lemma (A).]

For any terms s and t, let $s < t \equiv s \leq t \wedge s \neq t$, that is, $s < t$ is shorthand for $s \leq t \wedge s \neq t$.

Lemma (C). $T \vdash \forall xy[x < y \ \rightarrow \ \neg y \leq x]$ and $T \vdash \forall xy[\neg x \leq y \ \rightarrow \ y \neq x]$.

Problem 3

Prove Lemma (C) by giving T-derivations. [Hint: use Lemma (B).]

We extend the language $\mathcal{L}$ with a unary function symbol f, and we extend the theory T by the axioms

(A_4) $\forall x[x < f(x)]$

(A_5) $\forall xy[x < y \ \rightarrow \ f(x) < f(y)]$.

Problem 4

Prove that the axiom A_5 is independent of the other axioms; that is, prove $A_1, A_2, A_3, A_4 \not\vdash A_5$ and $A_1, A_2, A_3, A_4 \not\vdash \neg A_5$.

Let $f^0(t) \equiv t$ and $f^{n+1}(t) \equiv f(f^n(t))$.

Problem 5

Prove that we have $T \vdash t < f^\ell(t)$ for any $\mathcal{L}$-term t and any $\ell > 0$. [Hints: use induction on ℓ; use Lemma (C).]

Problem 6

Let $\mathfrak{A}$ be any model for T. Prove that there exists an $\mathcal{L}$-structure $\mathfrak{B}$ such that (i) $\mathfrak{A}$ and $\mathfrak{B}$ are elementary equivalent, and (ii) there exists $b_0, b_1, b_2, \ldots$ in the universe of $\mathfrak{B}$ such that

$$f^{\mathfrak{B}}(b_{i+1}) \ <^{\mathfrak{B}} \ f^{\mathfrak{B}}(b_i)$$

for any $i \in \mathbb{N}$.

END