UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in MAT-INF3600 — Mathematical logic.

Day of examination: Wednesday, December 2, 2015.

Examination hours: 9:00 – 13:00.

This problem set consists of 3 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Part I

Let R be a unary relation symbol, and let a and b be constant symbols. Let $\mathcal{L}$ be the language $\{a, b, R\}$. Let Σ_1 and Σ_2 be the sets of $\mathcal{L}$-formulas given by

$$\Sigma_1 \;=\; \{\, Ra \;,\; Rb \;,\; \forall xy[\,(Rx \wedge Ry) \rightarrow x = y\,]\} \;.$$

and

$$\Sigma_2 \;=\; \{\, \neg a = b \;,\; \forall xy[\,(Rx \wedge Ry) \rightarrow x = y\,]\} \;.$$

Problem 1

Give a Σ_1-deduction of $a = b$. Give a full deduction. Name all the logical axioms and inference rules involved in the deduction.

Problem 2

Give an $\mathcal{L}$-structure $\mathfrak{A}$ such that $\mathfrak{A} \models \Sigma_1$ and, moreover, the universe of $\mathfrak{A}$ contains exactly three elements.

Problem 3

Give a Σ_2-deduction of $(\neg Ra) \vee (\neg Rb)$. Give a full deduction. Name all the logical axioms and inference rules involved in the deduction.

Problem 4

Give an $\mathcal{L}$-structure $\mathfrak{B}$ such that $\mathfrak{B} \models \Sigma_2$ and, moreover, the universe of $\mathfrak{B}$ contains exactly three elements.

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Problem 5

Is the set $\Sigma_1 \cup \Sigma_2$ consistent? Justify your answer.

Problem 6

Does the set $\Sigma_1 \cup \Sigma_2$ have a model? Justify your answer.

Part II

Let $\mathcal{L}$ be any language, and let ϕ be an $\mathcal{L}$-formula with at least one free occurrence of the variable x. Let

$$(\exists^1 x)\phi \quad :\equiv \quad (\exists u)\phi^x_u \wedge (\forall y)(\forall z)[\,(\phi^x_y \wedge \phi^x_z) \to y = z\,]\,.$$

(So $(\exists^1 x)\phi$ is shorthand for the formula at the right hand side of $:\equiv$.)
Let A denote the universe of the $\mathcal{L}$-structure $\mathfrak{A}$, let $s : Vars \to A$ be an assignment, and let

$$A_{\phi,s} \quad = \quad \{\, a \;\mid\; a \in A \text{ and } \mathfrak{A} \models \phi\,[s[x|a]]\,\}$$

Now, we have

$$\mathfrak{A} \models (\exists^1 x)\phi[s] \text{ if and only if the set } A_{\phi,s} \text{ contains exactly one element.}$$

(We may state this informally: $(\exists^1 x)\phi(x)$ holds if and only if there is exactly one x such that $\phi(x)$ holds.)

Problem 7

Give an $\mathcal{L}$-formula $(\exists^2 x)\phi$ such that

$$\mathfrak{A} \models (\exists^2 x)\phi[s] \text{ if and only if the set } A_{\phi,s} \text{ contains exactly two elements.}$$

Let P and S be unary function symbols, and let 0 be a constant symbol. Let $\mathcal{L}_P$ be the language $\{0, P, S\}$. Let T be the $\mathcal{L}_P$-theory where we have the following non-logical axioms:

(T_1) $P(0) = 0$

(T_2) $(\exists^2 y)Py = 0$

(T_3) $(\forall x)[x \neq 0 \to (\exists^1 y)Py = x]$

(T_4) $(\forall x)PSx = x$

(T_5) $(\forall x)Sx \neq x$.

Problem 8

Give a model for T, that is, give an $\mathcal{L}_P$-structure $\mathfrak{A}$ such that $\mathfrak{A} \models T$.

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Problem 9

We have $T \vdash SP0 \neq PS0$. Sketch a T-deduction $SP0 \neq PS0$. Name all the non-logical axioms involved in the deduction.

Problem 10

Prove that the axiom (T5) is independent of the other axioms of T, that is, prove that

$$\{T_1, T_2, T_3, T_4\} \nvdash T_5 \quad \text{and} \quad \{T_1, T_2, T_3, T_4\} \nvdash \neg T_5 \ .$$

Problem 11

Prove that the axiom (T1) is independent of the other axioms of T, that is, prove that

$$\{T_2, T_3, T_4, T_5\} \nvdash T_1 \quad \text{and} \quad \{T_2, T_3, T_4, T_5\} \nvdash \neg T_1 \ .$$

Problem 12

Prove that any model for T is infinite.

Conjecture 1. Let $\mathcal{L}$ be any language. For any $\mathcal{L}$-formula ϕ with at least one free occurrence of the variable x, there exists an $\mathcal{L}$-formula $(\exists^{\infty} x)\phi$ such that

$$\mathfrak{A} \models (\exists^{\infty} x)\phi[s] \text{ if and only if the set } A_{\phi,s} \text{ is infinite.}$$

Problem 13

Prove that Conjecture 1 is wrong.

END