UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT-INF3600 — Mathematical logic.

Day of examination: Thursday, December 20, 2018.

Examination hours: 9:00 – 13:00.

This problem set consists of 3 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Part I

Theorem A. Let $\mathcal{L}$ be a first-order language, and let Σ be a set of $\mathcal{L}$-formulas. If every finite subset of Σ has a model, then Σ has a model.

Problem 1

State the Soundness Theorem for first-order logic. State the Completeness Theorem for first-order logic.

Problem 2

Use the Soundness and Completeness Theorem for first-order logic to prove Theorem A.

Part II

Let $\mathcal{L}$ be the first-order language $\{<, S\}$ where $<$ is a binary relation symbol and S is a unary function symbol. Let T be the $\mathcal{L}$-theory consisting of the non-logical axioms

(T_1) $\forall x[\ \neg x < x\ ]$

(T_2) $\forall xyz[\ (x < y \wedge y < z) \ \rightarrow\ x < z\ ]$

(T_3) $\forall x[\ x < Sx\ ]$.

Problem 3

Use the Soundness Theorem for first-order logic to prove that T_1 is independent of the other axioms, that is

$$T_2, T_3 \nvdash T_1 \quad \text{and} \quad T_2, T_3 \nvdash \neg T_1\ .$$

(Continued on page 2.)

Problem 4

Is T a consistent theory? Justify your answer.

Problem 5

Give a T-deduction of $\forall x[\, \neg Sx < Sx \,]$. Give a full deduction. Name all the logical axioms and inference rules involved in the deduction.

Problem 6

Sketch a T-deduction of $\forall x[\, \neg Sx = x \,]$. Name all the logical axioms and inference rules involved in the deduction.

Problem 7

Prove that any model for T is infinite.

Part III

The first-order language $\mathcal{L}_{NT}^{-}$ is $\{0, S, +, <\}$. We will use $\mathfrak{N}$ to denote the standard $\mathcal{L}_{NT}^{-}$-structure, that is, the structure where (i) the universe is $\mathbb{N}$, (ii) $0^{\mathfrak{N}}$ is 0, (iii) $S^{\mathfrak{N}}$ is the successor function, (iv) $+^{\mathfrak{N}}$ is addition and (v) $<^{\mathfrak{N}}$ is the standard strict ordering of $\mathbb{N}$ (note that we do not have multiplication and exponentiation). We will use $s \neq t$ and $s \not< t$ as shorthand for respectively $(\neg s = t)$ and $(\neg s < t)$.

The non-logical axioms of the $\mathcal{L}_{NT}^{-}$-theory N_0 are given by the axiom schemes (A1), (A2), (A3) and (A4):

$$s = t \tag{A1}$$

is an axiom when s and t are variable-free $\mathcal{L}_{NT}^{-}$-terms such that $\mathfrak{N} \models s = t$.

$$s \neq t \tag{A2}$$

is an axiom when s and t are variable-free $\mathcal{L}_{NT}^{-}$-terms such that $\mathfrak{N} \models s \neq t$.

$$s < t \tag{A3}$$

is an axiom when s and t are variable-free $\mathcal{L}_{NT}^{-}$-terms such that $\mathfrak{N} \models s < t$.

$$s \not< t \tag{A4}$$

is an axiom when s and t are variable-free $\mathcal{L}_{NT}^{-}$-terms such that $\mathfrak{N} \models s \not< t$.

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We define the *purely existential* formulas inductively.

- $s = t$, $s \neq t$, $s < t$, and $s \not< t$ are purely existential formulas when s and t are $\mathcal{L}_{NT}^{-}$-terms (so s and t may contain variables)
- $(\alpha \wedge \beta)$ and $(\alpha \vee \beta)$ are purely existential formulas when α and β are purely existential formulas
- $(\exists x)(\alpha)$ is a purely existential formula when α is a purely existential formula and x is a variable.

This completes the definition of the purely existential formulas.

Problem 8

Prove that we have

$$\mathfrak{N} \models \phi \;\;\Rightarrow\;\; N_0 \vdash \phi$$

for any purely existential sentence ϕ (a sentence is a formula with no free variables).

Problem 9

Prove that

$$N_0 \nvdash \forall x[\; x < \overline{6} \;\rightarrow\; (x = \overline{0} \vee x = \overline{1} \vee x = \overline{2} \vee x = \overline{3} \vee x = \overline{4} \vee x = \overline{5})\;] \;.$$

The Σ-formulas and the first-order theory N are known from our textbook. We restrict the definitions of Σ-formulas and N to the language $\mathcal{L}_{NT}^{-}$ (so N contains no axioms for multiplication and exponentiation).

Problem 10

Do we have

$$N \vdash \phi \;\;\Leftrightarrow\;\; N_0 \vdash \phi$$

for any purely existential sentence ϕ? Justify your answer.

Problem 11

Do we have

$$N \vdash \phi \;\;\Leftrightarrow\;\; N_0 \vdash \phi$$

for any Σ-sentence ϕ? Justify your answer.

Problem 12

Does there exist an $\mathcal{L}_{NT}^{-}$-structure $\mathfrak{A}$ such that $\mathfrak{A} \models N$ and $\mathfrak{A} \not\models N_0$? Justify your answer.

END