UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT-INF3600 — Mathematical logic.

Day of examination: Wednesday, December 18, 2019.

Examination hours: 14:30 – 18:30.

This problem set consists of 3 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Part I

Let P and Q be unary relation symbols. Let R be a binary relation symbol. Let c be a constant symbol. Let f be a unary function symbol. Furthermore, x and y denote variables.

Problem 1 *(weight 10 %)*

Let $\Sigma = \{ \neg Qc \, , \forall x[Px \rightarrow Qx] \}$. Give a full Σ-deduction of $\neg\forall x[Px]$. Name all the logical axioms and inference rules involved in the deduction.

Problem 2 *(weight 10 %)*

Let $\Sigma' = \{ \neg Qc \, , \forall x[Px \rightarrow Qx] \, , \forall x[Px] \}$. Is Σ' consistent? Does Σ' have a model? Give a brief justification of your answers.

Problem 3 *(weight 20 %)*

Twenty Questions: Answer each question with a YES or a NO (and nothing else). If you do not answer a question, your answer to that question will be considered as wrong.

1. Does $\forall x[Qx]$ follow tautologically from $\{ \forall x[Px] \rightarrow \forall x[Qx] \, , \forall x[Px] \}$?
2. Does $\forall x[Qx]$ follow logically from $\{ \forall x[Px] \rightarrow \forall x[Qx] \, , \forall x[Px] \}$?
3. Does Qc follow tautologically from $\{ \forall x[Px] \rightarrow \forall x[Qx] \, , \forall x[Px] \}$?
4. Does Qc follow logically from $\{ \forall x[Px] \rightarrow \forall x[Qx] \, , \forall x[Px] \}$?
5. Does $\forall x[Px \rightarrow Qx]$ follow logically from $\{ \forall x[Px] \rightarrow \forall x[Qx] \, , \forall x[Px] \}$?

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6. Does $\forall x[Px] \to \forall x[Qx]$ follow logically from $\{\ \forall x[Px \to Qx]\,,\ \forall x[Px]\ \}$?
7. Does $\forall x[Px \to Qx]$ follow logically from $\{\ \forall x[Px] \to \forall x[Qx]\ \}$?
8. Does $\forall x[Px] \to \forall x[Qx]$ follow logically from $\{\ \forall x[Px \to Qx]\ \}$?
9. Does $\exists y \forall x[Rxy]$ follow logically from $\{\ \forall x[Rxfx]\ \}$?
10. Does $\forall x \exists y[Rxy]$ follow logically from $\{\ \forall x[Rxfx]\ \}$?
11. Does $\exists y \forall x[Rxy]$ follow logically from $\{\ \forall x[Rxc]\ \}$?
12. Does $\forall x \exists y[Rxy]$ follow logically from $\{\ \forall x[Rxc]\ \}$?
13. Does $Qf(c)$ follow tautologically from $\{\ \forall x[Px \to Qx]\,,\ \forall x[Px] \to \forall x[Qx]\ \}$?
14. Does $Qf(c)$ follow logically from $\{\ \forall x[Px \to Qx]\,,\ \forall x[Px] \to \forall x[Qx]\ \}$?
15. Does $Pc \to \forall x[Qx]$ follow logically from $\{\ Pc \to Qx\ \}$?
16. Does $Px \to \forall x[Qx]$ follow logically from $\{\ Px \to Qx\ \}$?
17. Does $\exists x[Px] \to \forall x[Qx]$ follow logically from $\{\ Px \to \forall x[Qx]\ \}$?
18. Does $x = x$ follow logically from $\emptyset$?
19. Does $x = y$ follow logically from $\emptyset$?
20. Does $\neg x = y$ follow logically from $\emptyset$?

Part II

Let $\mathcal{L}$ be the first-order language $\{\preceq, f, c\}$ where $\preceq$ is a binary relation symbol, f is a binary function symbol and c is a constant symbol. Let T be the $\mathcal{L}$-theory consisting of the non-logical axioms

(T_1) $\forall xy[\ \neg\, c = f(x,y)\ ]$

(T_2) $\forall x_1 x_2 y_1 y_2[\ f(x_1,x_2) = f(y_1,y_2)\ \to\ (x_1 = y_1 \wedge x_2 = y_2)\ ]$

(T_3) $\forall x[\ x \preceq c\ \leftrightarrow\ x = c\ ]$

(T_4) $\forall x y_1 y_2[\ x \preceq f(y_1,y_2)\ \leftrightarrow\ (\ x = f(y_1,y_2)\ \vee\ x \preceq y_1\ \vee\ x \preceq y_2\ )\ ]$.

Problem 4 *(weight 10 %)*

Show that

$$T \vdash \neg f(c,c) = f(f(c,c),c)\ .$$

Sketch a formal deduction.

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Problem 5 *(weight 10 %)*

Show that

$$T \vdash \neg s = t \; .$$

for any variable-free $\mathcal{L}$-terms s, t where $s \neq t$ (so s and t are not syntactically equal). Use induction on the structure of s.

> **Lemma 1.** For any variable-free $\mathcal{L}$-terms s and t, we have $T \vdash s \preceq t$ or $T \vdash \neg s \preceq t$.

Problem 6 *(weight 10 %)*

Prove Lemma 1. Use induction on the structure of t.

Problem 7 *(weight 10 %)*

Let ϕ be a quantifier-free and variable-free $\mathcal{L}$-formula. Prove that we have $T \vdash \phi$ or $T \vdash \neg\phi$. Use Lemma 1.

Problem 8 *(weight 10 %)*

Do we have $T \vdash \forall x[\neg x = f(x, x)]$? Justify your answer.

We say that an $\mathcal{L}$-structure $\mathfrak{A}$ is *ill-founded* if its universe contains elements $a_0, a_1, a_2, \ldots$ such that $a_{i+1} \neq a_i$ and $a_{i+1} \preceq^{\mathfrak{A}} a_i$ (for all $i \in \mathbb{N}$).

Problem 9 *(weight 10 %)*

Explain why any consistent extension of T has an ill-founded model.

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