UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT-INF3600 — Mathematical logic.

Day of examination: Tuesday, December 1, 2021.

Examination hours: 15:00 – 19:00.

This problem set consists of 0 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

The weights might be adjusted.

PART I

Let P and Q be unary relation symbols, let c be a constant symbol and let $\mathcal{L}$ be the language $\{c, P, Q\}$. Furthermore, let $\Sigma = \{\, \forall x[\, Px \vee Qx \,]\,,\, \neg Qc \,\}$.

Problem 1 *(weight 10 %)*

Give a full Σ-deduction of Pc. Name all the logical axioms and inference rules involved in the deduction.

Problem 2 *(weight 10 %)*

Give a full Σ-deduction of $\exists x[\, Px \wedge \neg Qx \,]$. Name all the logical axioms and inference rules involved in the deduction.

Problem 3 *(weight 10 %)*

Give a finite $\mathcal{L}$-structure $\mathfrak{A}$ such that $\mathfrak{A} \models \Sigma$. Give an infinite $\mathcal{L}$-structure $\mathfrak{B}$ such that $\mathfrak{B} \models \Sigma$.

Problem 4 *(weight 10 %)*

Let ϕ be an arbitrary $\mathcal{L}$-sentence, and let $\Sigma_1 = \Sigma \cup \{\phi\}$ and $\Sigma_2 = \Sigma \cup \{\neg\phi\}$. Prove that at least one of Σ_1 and Σ_2 is consistent.

(Continued on page 2.)

PART II

Let 0 and 1 be constant symbols, let $<$ be a binary relation symbol, and let

$$\Sigma \;=\; \{\, \forall x[\, \neg x < x \,]\,,\; \forall xyz[\, x < y \wedge y < z \;\rightarrow\; x < z \,]\,,\; 0 < 1 \,\}\,.$$

Furthermore, let $t_1 \leq t_2 :\equiv t_1 < t_2 \vee t_1 = t_2$ and $t_1 \not\leq t_2 :\equiv \neg t_1 \leq t_2$ (for any terms t_1 and t_2, the formula $t_1 \leq t_2$ is shorthand for the formula $t_1 < t_2 \vee t_1 = t_2$, and the formula $t_1 \not\leq t_2$ is shorthand for the formula $\neg t_1 \leq t_2$).

Problem 5 *(weight 10 %)*

Ten Questions: Answer each question with a YES or a NO (and nothing else). If you do not answer a question, your answer to that question will be considered as wrong.

1. Do we have $\Sigma \models 1 \leq 0$?
2. Do we have $\Sigma \cup \{1 \leq 0\} \models \bot$?
3. Do we have $\Sigma \models 1 \not\leq 0$?
4. Do we have $\Sigma \models \forall x[0 \leq x \wedge x \leq 1]$?
5. Do we have $\Sigma \models \neg\forall x[0 \leq x \wedge x \leq 1]$?
6. Do we have $\Sigma \models \forall x[1 \leq x \rightarrow 0 < x]$?
7. Do we have $\Sigma \models \forall x[1 < x \rightarrow 0 < x]$?
8. Do we have $\Sigma \models \forall x[1 < x \leftrightarrow 0 < x]$?
9. Do we have $\Sigma \models \forall xy[x \not\leq y \rightarrow y < x]$?
10. Do we have $\Sigma \models \forall xy[x < y \vee x = y \vee y < x]$?

An $\mathcal{L}$-structure $\mathfrak{A}$ contains an *infinite descending chain* if the universe of $\mathfrak{A}$ contains a sequence of elements $a_0, a_1, a_2 \ldots$ such that $a_{i+1} <^{\mathfrak{A}} a_i$ (for all $i \in \mathbb{N}$). We say that an $\mathcal{L}$-structure is *ill-founded* if it contains an infinite descending chain.

Problem 6 *(weight 10 %)*

Teht Questions: Answer each question with a YES or a NO (and nothing else). If you do not answer a question, your answer to that question will be considered as wrong.

1. Is every infinite model of Σ ill-founded?
2. Are any two ill-founded models of Σ elementarily equivalent?
3. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two elementarily equivalent models of Σ. Will $\mathfrak{B}$ be ill-founded if $\mathfrak{A}$ is ill-founded?
4. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two isomorphic models of Σ. Will $\mathfrak{B}$ be ill-founded if $\mathfrak{A}$ is ill-founded?

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5. Does $\Sigma \cup \{\exists xy[x \not\leq y \wedge y \not\leq x]\}$ have a model that has exactly tree elements in its universe?
6. Does $\Sigma \cup \{\forall\exists y[x \not\leq y \wedge y \not\leq x]\}$ have a finite model?
7. Does $\Sigma \cup \{\forall\exists y[x \not\leq y \wedge y \not\leq x]\}$ have a model that has exactly tree elements in its universe?
8. Let $\mathfrak{A}$ be an ill-founded model of Σ. Does $\mathfrak{B} \prec \mathfrak{A}$ imply that $\mathfrak{B}$ is ill-founded?
9. Let $\mathfrak{A}$ be an ill-founded model of Σ. Does $\mathfrak{B} \prec \mathfrak{A}$ imply that $\mathfrak{B}$ is a model of Σ?
10. Let $\mathfrak{A}$ be an ill-founded model of Σ. Does $\mathfrak{A} \prec \mathfrak{B}$ imply that $\mathfrak{B}$ is ill-founded?

PART III

Let 0 be a constant symbol, let S be a unary function symbol, let $\cup$ and $\cap$ be a binary function symbols and, furthermore, let $\mathcal{L}$ be the first-order language $\{0, S, \cap, \cup\}$ and T be the $\mathcal{L}$-theory consisting of the non-logical axioms

(T_1) $\forall xy[\ Sx \neq 0\ ]$

(T_2) $\forall xy[\ Sx \neq Sy \ \rightarrow\ x = y\ ]$

(T_3) $\forall x[\ x \cup 0 = x\ ]$

(T_4) $\forall x[\ 0 \cup x = x\ ]$

(T_5) $\forall xy[\ Sx \cup Sy = S(x \cup y)\ ]$

(T_6) $\forall x[\ x \cap 0 = 0\ ]$

(T_7) $\forall x[\ 0 \cap x = 0\ ]$

(T_8) $\forall xy[\ Sx \cap Sy = S(x \cap y)\ ]$

For any $n \in \mathbb{N}$, define the *numeral* $\overline{n}$ by $\overline{0} = 0$ and $\overline{n+1} = S\overline{n}$

Theorem 1. For any $a, b, c \in \mathbb{N}$, we have

$$T \vdash \overline{a} \cap (\overline{b} \cup \overline{c}) \ = \ (\overline{a} \cap \overline{b}) \cup (\overline{a} \cap \overline{c})\ .$$

Theorem 2.

$$T \nvdash \forall xyz[\ x \cap (y \cup z) \ = \ (x \cap y) \cup (x \cap z)\ ]\ .$$

Problem 7 *(weight 20 %)*

Give a detailed proof of Theorem 1.

Problem 8 *(weight 20 %)*

Give a detailed proof of Theorem 2.

END

(Continued on page 4.)