UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT-INF3600 — Mathematical logic.

Day of examination: Monday, December 8, 2023.

Examination hours: 15:00 – 19:00.

This problem set consists of 3 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

The weights might be adjusted.

PART I *(weight 40 %)*

Let P and Q be unary relation symbols, let c be a constant symbol and let $\mathcal{L}$ be the language $\{c, P, Q\}$. Furthermore, let

$$\Sigma = \{\, Qc\,,\, \neg Pc\,,\, \exists x[\,\neg Qx\,]\,,\, \exists x[\,Px\,]\,\}\,.$$

Problem 1 *(weight 5 %)*

State the Soundness Theorem for first-order logic.

Problem 2 *(weight 5 %)*

Give an $\mathcal{L}$-structure $\mathfrak{A}$ such that $\mathfrak{A} \models \Sigma$ and $\mathfrak{A} \models \forall x[Px \vee Qx]$.

Problem 3 *(weight 5 %)*

Give an $\mathcal{L}$-structure $\mathfrak{B}$ such that $\mathfrak{A} \models \Sigma$ and $\mathfrak{A} \not\models \forall x[Px \vee Qx]$.

Problem 4 *(weight 5 %)*

Do we have $\Sigma \vdash \forall x[Px \vee Qx]$? Do we have $\Sigma \vdash \neg\forall x[Px \vee Qx]$? Justify your answers.

Problem 5 *(weight 5 %)*

What does it mean that a set of $\mathcal{L}$-formulas is consistent? Give the definition.

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Problem 6 *(weight 5 %)*

Is the set $\Sigma \cup \{\forall x[Px \vee Qx]\}$ consistent? Is the set $\Sigma \cup \{\neg\forall x[Px \vee Qx]\}$ consistent? Justify your answers.

Problem 7 *(weight 5 %)*

State the Completeness Theorem for first-order logic. State the Deduction Theorem for first-order logic.

Problem 8 *(weight 5 %)*

Let $\mathcal{L}$ be any first-order language and let Σ be a set of $\mathcal{L}$-sentences. Furthermore, let ψ be an $\mathcal{L}$-formula such that $\Sigma \models \psi$. Explain why there exists an $\mathcal{L}$-sentence ϕ such that

$$\vdash \phi \to \psi \quad \text{and} \quad \Sigma \vdash \phi \,.$$

PART II *(weight 35 %)*

Problem 9 *(weight 5 %)*

State the axiom scheme (E3).

Problem 10 *(weight 10 %)*

Let R be a unary relation symbol, and let x and y be variables. Prove that

$$\vdash Rx \to (x = y \to Ry)$$

Give a detailed deduction of the formula. Name alle the logical axioms and inference rules involved in the deduction.

Problem 11 *(weight 10 %)*

Let R be a unary relation symbol, and let $x_1, \ldots, x_n$ and y be variables. Prove that

$$\vdash (\ Rx_1 \wedge \ldots \wedge Rx_n\ ) \to (\ (\ x_1 = y \vee \ldots \vee x_n = y\ ) \to Ry\ )$$

Use induction on n.

Problem 12 *(weight 10 %)*

Let R be a unary relation symbol, and let $c_1, \ldots, c_n$ be constant symbols. Furthermore, let $\Sigma = \{\, \forall x[x = c_1 \vee \ldots \vee x = c_n] \,\}$. Prove that

$$\Sigma \vdash (\, Rc_1 \wedge \ldots \wedge Rc_n\,) \ \leftrightarrow\ \forall y[\, Ry\,] \,.$$

Sketch a Σ-deduction of the formula $(Rc_1 \wedge \ldots \wedge Rc_n) \leftrightarrow \forall y[Ry]$.

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PART III *(weight 25 %)*

Let $\mathcal{L}^-_{NT}$ be the language $\{0, S, +, \cdot\}$. We define the *numeral* $\overline{n}$ by $\overline{0} = 0$ and $\overline{n+1} = S\overline{n}$ (this is the standard definition of the numerals). Let N^- be the $\mathcal{L}^-_{NT}$-theory consisting of the non-logical axioms

(A_1) $\forall x[x = \overline{0} \vee x = \overline{1}]$

(A_2) $S\overline{1} = \overline{0}$

(A_3) $\forall x[\ x + 0 = x\ ]$

(A_4) $\forall xy[\ x + Sy = S(x + y)\ ]$

(A_5) $\forall x[\ x \cdot 0 = 0\ ]$

(A_6) $\forall x[\ x \cdot Sy = (x \cdot y) + x\ ]$.

Problem 13 *(weight 10 %)*

Prove that we have

$$N^- \vdash t = \overline{0} \quad \text{or} \quad N^- \vdash t = \overline{1}$$

for any variable-free $\mathcal{L}^-_{NT}$-term t.

Problem 14 *(weight 15 %)*

How many models has N^- up to isomorphism? (Give a number k such that any model for N^- will be isomorphic to one of these k models.)

Can you say something about how many models N^- has up to elementary equivalence?

Extend N^- to a complete theory, and prove that your extension is complete.

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