UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT-INF3600 — Mathematical logic.

Day of examination: Monday, December 20, 2024.

Examination hours: 15:00 – 19:00.

This problem set consists of 3 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

The weights might be adjusted.

Feel free to refer to the Soundness Theorem, the Completeness Theorem and the Deduction Theorem.

PART I *(weight 40 %)*

Let P, Q, R, S be unary relation symbols and let $\mathcal{L}$ be the language $\{P, Q, R, S\}$. Let

$$\Sigma = \{\ \forall x[Px \wedge (Px \to (Qx \vee Rx))],\ \forall x[Qx \to Sx],\ \forall x[Rx \to Sx]\ \}\ .$$

Problem 1

Show that $\Sigma \vdash Sx$. Give a full Σ-deduction of Sx. Name all the axioms and inference rules involved in the deduction.

Problem 2

Prove that $\Sigma \nvdash \forall x[Qx \wedge Rx]$.

Problem 3

Is the set $\Sigma \cup \{\forall x[Qx \wedge Rx]\}$ consistent? Justify your answer.

Let $\Sigma' = \Sigma \cup \{\ \exists x[\neg Sx]\ \}$.

Problem 4

Prove that $\Sigma' \vdash \bot$. Give a full Σ'-deduction of $\bot$. Name all the axioms and inference rules involved in the deduction.

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Problem 5

Let ϕ be any $\mathcal{L}$-sentence. Prove that at least one of the two sets

$$\Sigma \cup \{\phi\} \quad \text{and} \quad \Sigma \cup \{\neg\phi\}$$

is consistent.

Problem 6

Prove that there exists a set of $\mathcal{L}$-sentences Γ such that (i) $\Sigma \subseteq \Gamma$, (ii) Γ is consistent and (iii) for any $\mathcal{L}$-sentence ϕ, we have $\phi \in \Gamma$ or $\neg\phi \in \Gamma$. Will the sentence $\forall x[Sx]$ necessarily be in Γ? Will the sentence $\forall x[Qx \wedge Rx]$ necessarily be in Γ? Will the sentence $\neg\forall x[Qx \wedge Rx]$ necessarily be in Γ? Answer the questions briefly, but justify your answers.

PART II *(weight 30 %)*

Let a and b be constant symbols, let f be a binary function symbol and let $\mathcal{L}$ be the language $\{a, b, f\}$. We define the *clean terms* inductively:

- a is a clean term
- ft_1t_2 is a clean term if t_1 and t_2 are clean terms.

The set of $\mathcal{L}$-formulas Σ is given by

$$\Sigma \;=\; \{\; \neg t = b \mid t \text{ is a clean term} \;\}\;.$$

Problem 7

Give a finite $\mathcal{L}$-structure $\mathfrak{A}$ such that $\mathfrak{A} \models \Sigma$. Give an infinite $\mathcal{L}$-structure $\mathfrak{B}$ such that $\mathfrak{B} \models \Sigma$.

Problem 8

Give a set Δ_0 of $\mathcal{L}$-sentences such that (1) $\Sigma \cup \Delta_0$ is consistent and (2) any model for $\Sigma \cup \Delta_0$ is infinite. Prove that (1) and (2) holds.

Problem 9

Give a *finite* set Δ_1 of $\mathcal{L}$-sentences such that

1. Δ_1 is consistent
2. any model for Δ_1 is infinite

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3. for any formula ϕ, we have
$$\Sigma \vdash \phi \;\;\Rightarrow\;\; \Delta_1 \vdash \phi$$
(any formula that has a Σ-deduction will also have a Δ_1-deduction).

Prove that (1), (2) and (3) holds.

PART III *(weight 30 %)*

Let 0 be a constant symbol, let S be a unary function symbol, and let $<$ be a binary relation symbol. Let $\mathcal{L}_<$ be the language $\{0, S, <\}$. Let $N^<$ be the $\mathcal{L}_<$-theory consisting of the non-logical axioms

(A_1) $\forall x[\, \neg\, x < 0\,]$

(A_2) $\forall xy[\, x < Sy \leftrightarrow (x < y \vee x = y)\,]$.

Problem 10

Prove that $N^< \vdash 0 < S0$. Prove that $N^< \cup \{0 = S0\} \vdash \bot$. (Sketch the deductions. There is no need to give the details.)

We define the *basic formulas* inductively: an atomic $\mathcal{L}_<$-formula is a basic formula; $(\neg\alpha)$ is a basic formula if α is a basic formula; $(\alpha \vee \beta)$ is a basic formula if α and β are basic formulas.

A *basic sentence* is a basic formula with no free variables.

Problem 11

Prove that
$$N^< \vdash \phi \quad \text{or} \quad N^< \vdash \neg\phi$$
for any basic sentence ϕ.

We define the *bounded formulas* inductively: an atomic $\mathcal{L}_<$-formula is a bounded formula; $(\neg\alpha)$ is a bounded formula if α is a bounded formula; $(\alpha \vee \beta)$ is a bounded formula if α and β are bounded formulas; $\forall x[\, x < t \rightarrow \alpha\,]$ is a bounded formula if α is a bounded formula, x is a variable and t is an $\mathcal{L}_<$-term with no occurrences of x.

A *bounded sentence* is a bounded formula with no free variables.

Problem 12

Explain why we have
$$N^< \vdash \phi \quad \text{or} \quad N^< \vdash \neg\phi$$
for any bounded sentence ϕ. Try to sketch a proof.

END