Problem 1

The proof calculus in the textbook contains the two inference rules:

- $\langle\{\psi \to \phi\}, \psi \to (\forall x\phi)\rangle$
- $\langle\{\phi \to \psi\}, (\exists x\phi) \to \psi\rangle$.

To apply the rules, it is required that there are no free occurrences of x i ψ. Moreover, the proof calculus contains the axioms

- $(\forall x\phi) \to \phi_t^x$, if t is substitutable for x in ϕ
- $\phi_t^x \to (\exists x\phi)$, if t is substitutable for x in ϕ.

Problem a

Explain, briefly and informally, what it means that there are no free occurrences of x i ψ. Give an example of a first-order formula with free occurrences of x.

———————- Solution:

There is a free occurrence of x in the formula $\forall y[R(x, y)]$ since x does not occur inside the scope of a quantifier of the form $\forall x$ $(\exists x)$.

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Problem b

Explain, briefly and informally, what it means that t is substitutable for x in ϕ. Give an example of a first-order formula ϕ and a term t such that t *is not* substitutable for x in ϕ.

———————- Solution:

The term $f(y)$ is not substitutable for x in the formula $\forall y[R(x, y)]$ since the y in $f(y)$ will be bounded by a quantifier.

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Problem c

Prove that

$$\vdash \exists x[\,\phi_1 \wedge \phi_2\,] \to \exists x[\phi_1] \wedge \exists x[\phi_2]$$

by providing a derivation. The proof should not refer any lemmas or theorems from the textbook.

———————- Solution:

We have the following derivation (for arbitrary ϕ_1, ϕ_2):

1.	$\phi_1 \to \exists x[\phi_1]$	Q2, (x is always substitutable for x)
2.	$\phi_2 \to \exists x[\phi_2]$	Q2, (x is always substitutable for x)
3.	$\phi_1 \wedge \phi_2 \to \exists x[\phi_1] \wedge \exists x[\phi_2]$	1,2,PC
4.	$\exists x[\phi_1 \wedge \phi_2] \to \exists x[\phi_1] \wedge \exists x[\phi_2]$	3, QR,(no free occurrences of x in $\exists x[\phi_1] \wedge \exists x[\phi_2]$)

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Problem d

Let $n \in \mathbb{N}$ and la $\phi_1, \ldots, \phi_n$ be arbitrary first-order formulas. Prove by induction on n that we have

$$\vdash \exists x[\,\phi_1 \wedge \ldots \wedge \phi_n\,] \rightarrow \exists x[\phi_1] \wedge \ldots \wedge \exists x[\phi_n]$$

for every $n > 0$. The proof should not refer to any lemmas or theorems from the textbook.

———————————- Solution:

We have the following derivation (for arbitrary ϕ_1, ϕ_2):

We prove by induction on n that we can derive the formula

$$\phi_1 \wedge \ldots \wedge \phi_n \rightarrow \exists x[\phi_1] \wedge \ldots \wedge \exists x[\phi_n]\,. \qquad (*)$$

Induction start: When $n = 1$, the derivation is simply one application of the axiom Q2.

Induction step:

1.	$\phi_1 \wedge \ldots \wedge \phi_{n-1} \rightarrow \exists x[\phi_1] \wedge \ldots \wedge \exists x[\phi_{n-1}]$	ind. hyp.
2.	$\phi_n \rightarrow \exists x[\phi_n]$	Q2
3.	$\phi_1 \wedge \ldots \wedge \phi_n \rightarrow \exists x[\phi_1] \wedge \ldots \wedge \exists x[\phi_n]$	1,2,PC

This proves that (*) holds for any $n > 0$. The formula

$$\exists x[\,\phi_1 \wedge \ldots \wedge \phi_n\,] \rightarrow \exists x[\phi_1] \wedge \ldots \wedge \exists x[\phi_n]$$

can be derived from (*) by the rule QR.

Here is (a sketch of) an alternative induction proof:

By Problem **c**, we have

$$\vdash \exists x[\,(\phi_1 \wedge \ldots \wedge \phi_{n-1}) \wedge \phi_n\,] \rightarrow \exists x[\phi_1 \wedge \ldots \wedge \phi_{n-1}] \wedge \exists x[\phi_n]\,. \qquad (*)$$

By the ind. hyp. we have

$$\vdash \exists x[\,\phi_1 \wedge \ldots \wedge \phi_{n-1}\,] \rightarrow \exists x[\phi_1] \wedge \ldots \wedge \exists x[\phi_{n-1}] \qquad (**)$$

By PC, (*) and (**) we have

$$\vdash \exists x[\,\phi_1 \wedge \ldots \wedge \phi_n\,] \rightarrow \exists x[\phi_1] \wedge \ldots \wedge \exists x[\phi_n]$$

Problem e

Let ϕ_1 and ϕ_2 be arbitrary first-order formulas. Do we always have

$$\vdash \exists x[\phi_1] \wedge \exists x[\phi_2] \rightarrow \exists x[\,\phi_1 \wedge \phi_2\,]\,?$$

Justify your answer. You may refer to the Soundness Theorem for first-order logic.

———————- Solution:

The answer is no. Find formulas ϕ_1, ϕ_2 in some language $\mathcal{L}$ and an $\mathcal{L}$-structure $\mathfrak{A}$ such that

$$\mathfrak{A} \not\models \exists x[\phi_1] \wedge \exists x[\phi_2] \rightarrow \exists x[\,\phi_1 \wedge \phi_2\,] \qquad (*)$$

It follows from (*) and the Soundness Theorem for first-order logic that

$$\not\vdash \exists x[\phi_1] \wedge \exists x[\phi_2] \rightarrow \exists x[\,\phi_1 \wedge \phi_2\,]\,.$$

(The solution of Problem **2b** gives such a structure $\mathfrak{A}$.)

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Problem 2

Let P and Q be unary relation symbols, let S be a unary function symbol, and let 0 be a constant symbol. Let $\mathcal{L}$ be the first-order language $\{0, S, P, Q\}$, and let Σ_0 be the $\mathcal{L}$-theory consisting of the following three non-logical axioms:

- $\forall x[\neg 0 = Sx]$
- $\forall xy[Sx = Sy \rightarrow x = y]$
- $\exists x[Px] \wedge \exists x[Qx] \rightarrow \exists x[\,Px \wedge Qx\,]$.

Problem a

Let $\mathfrak{A}$ be a structure for Σ_0, i.e. $\mathfrak{A} \models \Sigma_0$. Prove that the universe of $\mathfrak{A}$ is infinite.

———————- Solution:

Let A denote the universe of $\mathfrak{A}$, and let $|X|$ denote the cardinality of the set X. Assume that the universe if finite, i.e. $|A| = n$ for some $n \in \mathbb{N}$. By the first axiom we have

$$S^{\mathfrak{A}} : A \rightarrow A \setminus \{0^{\mathfrak{A}}\}$$

and $|A \setminus \{0^{\mathfrak{A}}\}| = n - 1$. By the second axiom $S^{\mathfrak{A}}$ is an injection. It is not possible to have an injection from a set containing n elements into a set containing $n-1$. Thus, the assumption that A is finite leads to a contradiction, and we conclude that A is infinite.

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Problem b

Give to $\mathcal{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$ fulfilling the following requirements:

- the universe of $\mathfrak{A}$ is $\mathbb{N}$
- the universe of $\mathfrak{B}$ is $\mathbb{N}$
- $\mathfrak{A} \models \Sigma_0$ and $\mathfrak{B} \not\models \Sigma_0$.

($\mathbb{N}$ denotes the set of natural numbers, i.e. $\mathbb{N} = \{0, 1, 2, \ldots\}$.)

———————- Solution:

Let $0^{\mathfrak{A}} = 0^{\mathfrak{B}} = 0$ and $S^{\mathfrak{A}}(x) = S^{\mathfrak{B}}(x) = x + 1$. Let

$$P^{\mathfrak{A}} = P^{\mathfrak{B}} = \{x \mid x \in \mathbb{N} \text{ and } x \text{ is even}\} .$$

Let

$$Q^{\mathfrak{A}} = \{x \mid x \in \mathbb{N} \text{ and } x \text{ is a prime}\}$$

and

$$Q^{\mathfrak{B}} = \{x \mid x \in \mathbb{N} \text{ and } x \text{ is odd}\} .$$

Now, both even and odd numbers exist, but there does not exists a particular number that is both even and odd. Hence, $\mathfrak{B} \not\models \Sigma_0$. In contrast, we have $\mathfrak{A} \models \Sigma_0$ as 2 is an even prime.

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Let

- $\Sigma_1 \;=\; \Sigma_0 \cup \{\, \forall x[\, Px \leftrightarrow \neg Qx \,] \,\}$
- $\Sigma_2 \;=\; \Sigma_1 \cup \{\, \exists x[\, Px \,] \wedge \exists x[\, Qx \,] \,\}.$

Problem c

What does it mean that a set of $\mathcal{L}$-formulas is inconsistent?

———————- Solution:

A set Γ of $\mathcal{L}$-formulas is inconsistent if $\Gamma \vdash \bot$ where $\bot \equiv \forall x[x = x] \wedge \neg\forall x[x = x]$.

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Problem d

Prove that Σ_2 is an inconsistent theory.

———————- Solution:

For any model $\mathfrak{A}$ such that $\mathfrak{A} \models \Sigma_1$, we have $P^{\mathfrak{A}} \cap Q^{\mathfrak{A}} = \emptyset$. Thus, we also have $P^{\mathfrak{A}} \cap Q^{\mathfrak{A}} = \emptyset$ for any $\mathfrak{A}$ such that $\mathfrak{A} \models \Sigma_2$.

Assume $\mathfrak{A} \models \Sigma_2$. Since

$$\forall x[\, Px \leftrightarrow \neg Qx \,] \;\in\; \Sigma_2$$

we have $P^{\mathfrak{A}} \cap Q^{\mathfrak{A}} = \emptyset$. Since

$$\exists x[\, Px \,] \wedge \exists x[\, Qx \,] \;\in\; \Sigma_2$$

we have $P^{\mathfrak{A}} \cap Q^{\mathfrak{A}} \neq \emptyset$. This is a contradiction, and we conclude that Σ_2 does not have a model. By the Completeness Theorem for first-order logic (the "Henkin" version), Σ_2 is an inconsistent theory.

It is also possible to solve this problem by providing a Σ_2-derivation of $\bot$.

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Problem e

Do we have $\Sigma_1 \vdash \exists x[Px]$? Do we have $\Sigma_1 \vdash \neg\exists x[Px]$? Justify your answer.

———————- Solution:

We define two models $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathfrak{A} \models \Sigma_1$ and $\mathfrak{B} \models \Sigma_1$. Both models have the same universe: $\mathbb{N}$. Let $0^{\mathfrak{A}} = 0^{\mathfrak{B}} = 0$ and $S^{\mathfrak{A}}(x) = S^{\mathfrak{B}}(x) = x + 1$. Let $P^{\mathfrak{A}} = \mathbb{N}$ and $Q^{\mathfrak{A}} = \emptyset$. $P^{\mathfrak{B}} = \emptyset$ and $Q^{\mathfrak{B}} = \mathbb{N}$.

Now, we have $\mathfrak{A} \models \Sigma_1$ and $\mathfrak{A} \not\models \neg\exists x[Px]$. Thus, $\Sigma_1 \not\models \neg\exists x[Px]$. Thus, by the Soundness Theorem for first-order logic, we have $\Sigma_1 \nvdash \neg\exists x[Px]$.

Furthermore, we have $\mathfrak{B} \models \Sigma_1$ and $\mathfrak{B} \not\models \exists x[Px]$. Thus, $\Sigma_1 \not\models \exists x[Px]$. Thus, by the Soundness Theorem for first-order logic, we have $\Sigma_1 \nvdash \exists x[Px]$.

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Let $\Sigma_3 = \Sigma_1 \cup \{\, \forall x[Px] \,\}$.

Problem f

Suppose we are working with first-order logic *without* equality, that is, our language does not contain the relation symbol $=$. Furthermore, suppose we have $\mathcal{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathfrak{A} \models \Sigma_3$ and $\mathfrak{B} \models \Sigma_3$. Prove that

$$\mathfrak{A} \models \phi \text{ if and only if } \mathfrak{B} \models \phi$$

for every $\mathcal{L}$-formula ϕ.

———————- Solution:

Let A denote the universe of $\mathfrak{A}$.

Observation: If $\mathfrak{A} \models \Sigma_3$, then $P^{\mathfrak{A}} = A$ and $Q^{\mathfrak{A}} = \emptyset$.

> **(Claim I)** For any ϕ not containing the equality symbol, we have
>
> $$\mathfrak{A} \models \phi[s_0] \text{ for some assignment } s_0$$
>
> if and only if
>
> $$\mathfrak{A} \models \phi[s] \text{ for every assignment } s$$

Prove (Claim I) by induction over the structure of ϕ.

> **(Claim II)** Let $\mathfrak{A}$ and $\mathfrak{B}$ be arbitrary models for Σ_3. For any ϕ not containing the equality symbol, we have
>
> $$\mathfrak{A} \models \phi \;\; \Leftrightarrow \;\; \mathfrak{B} \models \phi$$
>
> and
>
> $$\mathfrak{A} \models \neg\phi \;\; \Leftrightarrow \;\; \mathfrak{B} \models \neg\phi \,.$$

Prove (Claim II) by induction over the structure of ϕ. Use (Claim I) in the proof.

We do the induction step in the case when ϕ os of the form $\phi \equiv \forall x\psi$. Recall that the definition says that

$$\mathfrak{A} \models \phi \;\; \Leftrightarrow \;\; \mathfrak{A} \models \phi[s] \text{ for all assignments } s$$

We have

$$\begin{array}{rcll}
\mathfrak{A} \models \forall x\psi & \Leftrightarrow & \mathfrak{A} \models \forall x\psi[s] \text{ for all assignments } s & \text{def.} \\
& \Leftrightarrow & \mathfrak{A} \models \psi[s] \text{ for all assignments } s & \text{def. of } \forall \\
& \Leftrightarrow & \mathfrak{B} \models \psi[s] \text{ for all assignments } s & \text{ind. hyp.} \\
& \Leftrightarrow & \mathfrak{B} \models \forall x\psi[s] \text{ for all assignments } s & \text{def. of } \forall
\end{array}$$

Furthermore, we have

$$\begin{array}{rcll}
\mathfrak{A} \models \neg\forall x\psi & \Leftrightarrow & \mathfrak{A} \models \neg\forall x\psi[s] \text{ for all assignments } s & \text{def.} \\
& \Leftrightarrow & \mathfrak{A} \not\models \forall x\psi[s] \text{ for all assignments } s & \text{def. of a model} \\
& \Leftrightarrow & \mathfrak{A} \not\models \forall x\psi[s_0] \text{ for some assignment } s_0 & \text{(Claim I)} \\
& \Leftrightarrow & \mathfrak{A} \models \neg\psi[s_0] \text{ for some assignment } s_0 & \text{def. of a model} \\
& \Leftrightarrow & \mathfrak{A} \models \neg\psi[s] \text{ for all assignments } s & \text{(Claim I)} \\
& \Leftrightarrow & \mathfrak{B} \models \neg\psi[s] \text{ for all assignments } s & \text{ind. hyp.} \\
& \Leftrightarrow & \mathfrak{B} \models \neg\psi[s_0] \text{ for some assignment } s_0 & \text{(Claim I)} \\
& \Leftrightarrow & \mathfrak{B} \not\models \forall x\psi[s_0] \text{ for some assignment } s_0 & \text{def. of a model} \\
& \Leftrightarrow & \mathfrak{B} \not\models \forall x\psi[s] \text{ for all assignments } s & \text{(Claim I)} \\
& \Leftrightarrow & \mathfrak{B} \models \neg\forall x\psi[s] \text{ for all assignments } s & \text{def. of a model}
\end{array}$$

Problem g

Suppose we are working with first-order logic *with* equality. Do there exist structures $\mathfrak{A}$ and $\mathfrak{B}$ and an $\mathcal{L}$-formula ϕ such that

$$\mathfrak{A} \models \Sigma_3 \quad \text{and} \quad \mathfrak{A} \models \phi$$

and

$$\mathfrak{B} \models \Sigma_3 \quad \text{and} \quad \mathfrak{B} \models \neg\phi \, ?$$

If the answer is yes, provide such structures. If the answer is no, explain why.

———————- Solution:

The answer is yes.

We give the structure $\mathfrak{A}$. Let $\mathbb{N} \cup \{\omega\}$ be the universe of $\mathfrak{A}$. Let $0^{\mathfrak{A}} = 0$ and

$$S^{\mathfrak{A}}(x) = \begin{cases} x+1 & \text{if } x \in \mathbb{N} \\ x & \text{if } x = \omega. \end{cases}$$

Let $P^{\mathfrak{A}} = \mathbb{N} \cup \{\omega\}$ and $Q^{\mathfrak{A}} = \emptyset$.

We give the structure $\mathfrak{B}$. Let $\mathbb{N}$ be the universe of $\mathfrak{B}$. Let $0^{\mathfrak{B}} = 0$ and $S^{\mathfrak{B}}(x) = x + 1$ (standard successor). Let $P^{\mathfrak{B}} = \mathbb{N}$ and $Q^{\mathfrak{B}} = \emptyset$.

We have $\mathfrak{A} \models \Sigma_3$ and $\mathfrak{B} \models \Sigma_3$. Furthermore, we have

$$\mathfrak{A} \models \exists x[x = S(x)] \quad \text{and} \quad \mathfrak{B} \models \neg\exists x[x = S(x)] \, .$$

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Problem 3

Problem a

What does it mean that a set is recursive? Give the definition from the textbook. Give a short answer.

———————- Solution:

A set $A \subseteq \mathbb{N}^k$ is said to be *recursive* if there is an $\mathcal{L}_{NT}$-formula $\phi(x_1, \ldots, x_k)$ such that

- $N \vdash \phi(\overline{a}_1, \ldots, \overline{a}_k)$ for every $\langle a_1, \ldots, a_k \rangle \in A$
- $N \vdash \neg\phi(\overline{a}_1, \ldots, \overline{a}_k)$ for every $\langle a_1, \ldots, a_k \rangle \notin A$.

($\mathcal{L}_{NT} = \{0, S, +, \cdot, E, <\}$, and N is a particular set of axioms given in the textbook.)

Problem b

What does it mean that a first-order theory is axiomatizable? What does it mean that a first-order theory is complete? Give short answers.

———————————- Solution:

Let Σ be a set of first-order axioms over some language $\mathcal{L}$, and let $\lceil\phi\rceil$ denote the Gödel number of the $\mathcal{L}$-formula ϕ. The theory Σ is *axiomatizable* if the set $\{\lceil\phi\rceil \mid \phi \in \Sigma\}$ is recursive. The theory Σ is *complete* if we have

$$\Sigma \vdash \phi \quad \text{or} \quad \Sigma \vdash \neg\phi$$

for any $\mathcal{L}$-formula ϕ.

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Problem c

Is the theory Σ_3 from Problem 2 an axiomatizable, consistent and complete theory? Does it matter in this respect if the language contains equality? Justify your answer.