The following theorem is known from Leary's book.

> **Theorem (A).** Let $\mathcal{L}$ be a first-order language. For any $\mathcal{L}$-theory T, any variable x, and any $\mathcal{L}$-formula ϕ, we have
>
> $$T \vdash \forall x \phi \quad \Leftrightarrow \quad T \vdash \phi \, .$$

Problem 1

Prove Theorem (A).

———————— Solution:

This is a standard proof. See Leary's book.

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Let 0 be a constant symbol, let $'$ be a unary function symbol, and let $<$ be a binary relation symbol. Let $\mathcal{L}$ be the first-order language $\{0, ', <\}$, and let T be the $\mathcal{L}$-theory consisting of the following non-logical axioms:

(T1) $\forall x \, [\, \neg x < x \,]$

(T2) $\forall xyz \, [\, x < y \ \wedge \ y < z \ \rightarrow x < z \,]$

(T3) $\forall xy \, [\, x < y \ \rightarrow x' < y' \,]$

(T4) $0 < 0'$

Problem 2

Give a T-derivation of $0' < 0''$. Name the logical and the non-logical axioms involved in the derivation.

———————— Solution:

We have the following T-derivation:

1.	$\forall xy \, [\, x < y \ \rightarrow \ x' < y' \,]$	T3
2.	$\forall xy \, [\, x < y \ \rightarrow x' < y' \,] \ \rightarrow \ \forall y \, [\, 0 < y \ \rightarrow \ 0' < y' \,]$	Q1
3.	$\forall y \, [\, 0 < y \ \rightarrow 0' < y' \,] \ \rightarrow \ [\, 0 < 0' \ \rightarrow \ 0' < 0'' \,]$	Q1
4.	$0 < 0' \ \rightarrow \ 0' < 0''$	1,2,3, PC
5.	$0 < 0'$	T4
6.	$0' < 0''$	4,5,PC

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Problem 3

Give a T-derivation of $\forall xy[x < y \rightarrow \neg x = y]$. Name the logical and the non-logical axioms involved in the derivation. You may apply Theorem (A). [Hint: You will need the logical axiom $x = x \ \wedge \ x = y \ \rightarrow \ (x < y \ \rightarrow \ x < x)$ (E3).]

——————————— Solution:

We have the following T-derivation:

1.	$x = x \ \wedge \ x = y \ \rightarrow \ (x < y \ \rightarrow \ x < x)$	E3
2.	$x = x$	E1
3.	$x = y \ \wedge \ x < y \ \rightarrow \ x < x$	1,2, PC
4.	$\forall x \, [\, \neg x < x \,]$	T1
5.	$\neg x < x$	4, (A)
6.	$x < y \ \rightarrow \ \neg x = y$	3,5,PC
7.	$\forall xy[x < y \rightarrow \neg x = y]$	6, (A)

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For $n \in \mathbb{N}$, we define the $\mathcal{L}$-term $0^{[n]}$ by $0^{[0]} \equiv 0$ and $0^{[m+1]} \equiv 0^{[m]\prime}$.

Problem 4

Prove that

$$m < n \ \ \Rightarrow \ \ T \vdash \neg \, 0^{[m]} = 0^{[n]} \, .$$

[Hint: Use induction on n to prove that we have $T \vdash 0^{[i]} < 0^{[n+1]}$ for any $i \leq n$.]

——————————— Solution:

We prove

$$T \vdash 0^{[i]} < 0^{[n+1]} \text{ for any } i \leq n \qquad (*)$$

by induction on n.

Induction start. We have $T \vdash 0 < 0'$ by (T4). Hence, (*) holds when $n = 0$.

Induction step. Assume by induction hypothesis that (*) holds. We will prove that we have $T \vdash 0^{[i]} < 0^{[n+2]}$ for any $i \leq n + 1$.

First, we prove

$$T \vdash 0^{[n+1]} < 0^{[n+2]} \qquad (\dagger)$$

by giving a T-derivation.

1.	$\forall xy \, [\, x < y \ \rightarrow x' < y' \,]$	T3
2.	$\forall xy \, [\, x < y \ \rightarrow x' < y' \,] \ \rightarrow \ \forall y \, [\, 0^{[n]} < y \ \rightarrow 0^{[n+1]} < y' \,]$	Q1
3.	$\forall y \, [\, 0^{[n]} < y \ \rightarrow 0^{[n+1]} < y' \,] \ \rightarrow \ [\, 0^{[n]} < 0^{[n+1]} \ \rightarrow \ 0^{[n+1]} < 0^{[n+2]} \,]$	Q1
4.	$0^{[n]} < 0^{[n+1]} \ \rightarrow \ 0^{[n+1]} < 0^{[n+2]}$	1,2,3,PC
5.	$0^{[n]} < 0^{[n+1]}$	(*)
6.	$0^{[n+1]} < 0^{[n+2]}$	4,5,PC

This proves (†). Now, fix an arbitrary k such that $k \leq n$. We will prove that $T \vdash 0^{[k]} < 0^{[n+2]}$ by giving a T-derivation.

1.	$\forall xyz\,[\,x < y \;\wedge\; y < z \;\rightarrow x < z\,]$	T3
2.	$0^{[k]} < 0^{[n+1]} \;\wedge\; 0^{[n+1]} < 0^{[n+2]} \;\rightarrow 0^{[k]} < 0^{[n+2]}\,]$	1,Q1,PC
3.	$0^{[k]} < 0^{[n+1]}$	(*)
4.	$0^{[n+1]} < 0^{[n+2]}$	(†)
5.	$0^{[k]} < 0^{[n+2]}$	2,3,4,PC

This completes the induction steps.

We have now proved (*). The following T-derivation shows that we have $T \vdash \neg\, 0^{[m]} = 0^{[n]}$ whenever $m < n$.

1.	$\forall xy[x < y \rightarrow \neg x = y]$	Problem 3
2.	$0^{[m]} < 0^{[n]} \;\rightarrow\; \neg 0^{[m]} = 0^{[n]}$	1,QR,PC
3.	$0^{[m]} < 0^{[n]}$	(*)
4.	$\neg 0^{[m]} = 0^{[n]}$	2,3,PC

Let T^- denote T without the axiom (T4), that is, T^- is the first-order theory consisting of the axioms (T1), (T2) and (T3).

Problem 5

Give a model $\mathfrak{A}$ for T^- such that the universe of $\mathfrak{A}$ contains exactly 17 elements.

——————————— Solution:

Let $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 17\}$.[1] Let $a'^{\mathfrak{A}} = a$ for any $a \in A$. Let $<^{\mathfrak{A}} = \emptyset$.

Problem 6

Explain why (T4) is independent of the other axioms in T, that is, explain why $T^- \nvdash (T4)$, and explain why $T^- \nvdash \neg(T4)$.

——————————— Solution:

The model $\mathfrak{A}$ from Problem 5 has the following properties: $\mathfrak{A} \models T^-$ and $\mathfrak{A} \not\models (T4)$. Hence, by the Soundness Theorem for first-order logic, we have $T^- \nvdash (T4)$.

The model $\mathfrak{N}$ given below has the following properties: $\mathfrak{N} \models T^-$ and $\mathfrak{A} \not\models \neg(T4)$. Hence, by the Soundness Theorem for first-order logic, we have $T^- \nvdash \neg(T4)$.

We define the $\mathcal{L}$-structure $\mathfrak{N}$: The universe N is the set of natural numbers, that is, $\{0, 1, 2, \ldots\}$. The relation $<^{\mathfrak{N}}$ is the standard strict ordering of the natural numbers. The function $'^{\mathfrak{N}}$ is the successor function, that is, $n'^{\mathfrak{N}} = n + 1$ for any $n \in N$. Finally, $0^{\mathfrak{N}} = 0$.

[1] There is some sick nerd humor here.

Recall that an $\mathcal{L}$-structure $\mathfrak{A}$ is *elementary equivalent* to an $\mathcal{L}$-structure $\mathfrak{B}$ if, and only if, we have

$$\mathfrak{A} \models \phi \quad \Leftrightarrow \quad \mathfrak{B} \models \phi$$

for any $\mathcal{L}$-sentence ϕ.

Problem 7

Give a model $\mathfrak{A}$ for T that is not elementary equivalent to $\mathfrak{N}$. Give an $\mathcal{L}$-sentence ϕ such that $\mathfrak{A} \models \phi$ and $\mathfrak{N} \models \neg\phi$.

——————————— Solution:

Let the universe A be the set of rational numbers. Let $<^{\mathfrak{A}}$ be the standard strict ordering of the rational numbers. Let $a'^{\mathfrak{A}} = a + 1$ for any $a \in A$. Finally, let $0^{\mathfrak{A}} = 0$. Let

$$\phi \;\equiv\; \forall xy\,[\,x < y \;\rightarrow\; \exists z\,[\,x < z \,\wedge\, z < y\,]\,]\,.$$

(This problem has a lot of natural solutions. The solution given here is just meant as an example.)

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Problem 8

Construct a countable model $\mathfrak{A}$ for T that is elementary equivalent, but not isomorphic, to $\mathfrak{N}$. [Hint: Use the Compactness Theorem.]

——————————— Solution:

Let $Th(\mathfrak{N}) = \{\phi \mid \mathfrak{N} \models \phi\}$.

Let $\mathcal{L}_c$ be $\mathcal{L}$ extended with a fresh constant c.

For any $n \in \mathbb{N}$, let $\Gamma_n = Th(\mathfrak{N}) \cup \{0^{[j]} < c \mid j \leq n\}$.

Let $\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n$.

Let $\mathfrak{N}_\ell$ be the $\mathcal{L}_c$-structure we achieve by extending $\mathfrak{N}$ by the interpretation $c^{\mathfrak{N}_\ell} = \ell + 1$.

Then, $\mathfrak{N}_\ell \models \Gamma_\ell$.

(Claim) Γ has a countable model.

We prove the claim: Let $\Gamma_\bullet$ be an arbitrary finite subset of Γ. Then, we have $\Gamma_\bullet \subset \Gamma_\ell$ for all sufficiently large $\ell \in \mathbb{N}$. Thus, fix a sufficiently large ℓ, and we have $\mathfrak{N}_\ell \models \Gamma_\bullet$. This proves that any finite subset of Γ has a model. By the Compactness Theorem, we conclude that Γ has a countable model. This completes the proof of the claim.

Let $\mathfrak{A}_c \models \Gamma$. (Now, $\mathfrak{A}$ can be any countable model of Γ, and by the claim we now that such models exist.)

Now, $\mathfrak{A}_c$ is an $\mathcal{L}$-structure. Let $\mathfrak{A}$ be $\mathfrak{A}_c$ restricted to the language $\mathcal{L}$.

For any $\mathcal{L}$-sentence ϕ, we have

$$\mathfrak{N} \models \phi \;\Rightarrow\; \phi \in \Gamma \;\Rightarrow\; \mathfrak{A} \models \phi$$

and

$$\mathfrak{N} \not\models \phi \;\Rightarrow\; \mathfrak{N} \models \neg\phi \;\Rightarrow\; (\neg\phi) \in \Gamma \;\Rightarrow\; \mathfrak{A} \models \neg\phi \;\Rightarrow\; \mathfrak{A} \not\models \phi\,.$$

Thus, $\mathfrak{N} \models \phi$ iff $\mathfrak{A} \models \phi$. This proves that $\mathfrak{N}$ and $\mathfrak{A}$ are elementary equivalent.

Assume that $\mathfrak{N}$ and $\mathfrak{A}$ are isomorphic. (This assumption leads to a contradiction.) Then, we have a bijection $\imath : N \to A$ such that

$$\imath(0^{\mathfrak{N}}) = 0^{\mathfrak{A}} \quad \text{and} \quad \imath(x^{(\prime\mathfrak{N})}) = \imath(x)^{(\prime\mathfrak{A})} \ .$$

By a straight forward induction, we can prove that we have

$$\imath(0^{[\ell]\mathfrak{N}}) \ = \ 0^{[\ell]\mathfrak{A}}$$

for any ℓ. This contradicts that $\imath$ is a bijection, as

$$N \ = \ \{\, a \mid a = 0^{[\ell]\mathfrak{N}} \text{ and } \ell \in \mathbb{N} \,\}$$

whereas

$$A \ \subseteq \ \{\, a \mid a = 0^{[\ell]\mathfrak{A}} \text{ and } \ell \in \mathbb{N}\} \ \cup \ \{c^{\mathfrak{A}_c}\} \ .$$

The three remaining problems are related to the material in Lewis's & Padimitriou's textbook. No written solutions will be worked out for these problems.

Problem 9

Give a standard definition of the primitive recursive functions. You can e.g. give the definition from Lewis's & Padimitriou's book.

We define the bounded quantifiers in the standard way, that is $\forall x < y[\phi] \equiv \forall x[x < y \to \phi]$ and $\exists x < y[\phi] \equiv \exists x[x < y \wedge \phi]$. A Δ_0-formula is an $\mathcal{L}$-formula where all quantifiers are bounded.

> **Theorem (B).** For any Δ_0-formula $\phi(x_1, \ldots, x_n)$, there exists a primitive recursive function f_ϕ such that
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> - $\mathfrak{N} \models \phi(0^{[a_1]}, \ldots, 0^{[a_n]}) \quad \Leftrightarrow \quad f_\phi(a_1, \ldots, a_n) = 0$
> - $\mathfrak{N} \models \neg\phi(0^{[a_1]}, \ldots, 0^{[a_n]}) \quad \Leftrightarrow \quad f_\phi(a_1, \ldots, a_n) = 1.$

Problem 10

Prove Theorem (B). You can refer to well-known facts about the primitive recursive functions, e.g., to facts stated in Lewis's & Padimitriou's book.

Problem 11

Do you think Theorem (B) holds for $\mathcal{L}$-formulas in general, that is, is the theorem still true if we replace "any Δ_0-formula $\phi(x_1, \ldots, x_n)$" by "any $\mathcal{L}$-formula $\phi(x_1, \ldots, x_n)$"? Answer short, but justify your answer.