UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT-INF3600 — Mathematical logic.

Day of examination: Wednesday, December 3, 2014.

Examination hours: 9:00 – 13:00.

This problem set consists of 7 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Part I

Let P and Q be unary relation symbols, and let a be a constant symbol. Let $\mathcal{L}$ be the language $\{P, Q, a\}$. Furthermore, let

- $\Sigma_1 = \{\, \forall x[Px \to Qx]\,,\, Pa\,\}$
- $\Sigma_2 = \{\, \forall x[Px] \to \forall x[Qx]\,,\, Pa\,\}$
- $\Sigma_3 = \{\, \neg(\forall x[Px] \to \forall x[Qx])\,,\, \neg Qa\,\}$
- $\Sigma_4 = \{\, \neg(\forall x[Px] \to \forall x[Qx])\,,\, \neg Pa\,\}$.

Problem 1

Answer each of the following questions by a YES or a NO. There is no need to justify your answer.

(a) Does Qa follow tautologically from Σ_1?

(b) Does Qa follow logically from Σ_1?

(c) Does Qa follow tautologically from Σ_2?

(d) Does Qa follow logically from Σ_2?

(e) Does $\forall x[Px]$ follow tautologically from Σ_3?

(f) Does $\forall x[Px]$ follow logically from Σ_3?

(g) Does Pa follow tautologically from Σ_4?

(h) Does Pa follow logically from Σ_4?

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———————————— Solution:

(a) Does Qa follow tautologically from Σ_1? NO

(b) Does Qa follow logically from Σ_1? YES

(c) Does Qa follow tautologically from Σ_2? NO

(d) Does Qa follow logically from Σ_2? NO

(e) Does $\forall x[Px]$ follow tautologically from Σ_3? YES

(f) Does $\forall x[Px]$ follow logically from Σ_3? YES

(g) Does Pa follow tautologically from Σ_4? NO

(h) Does Pa follow logically from Σ_4? YES

Note that Σ_4 is logically inconsistent (but not tautologically inconsistent).
———————————— End Solution

Let $\Sigma_5 = \{\forall x[Px \to Qx], \neg Qa\}$.

Problem 2

Give a Σ_5-deduction of $\neg Pa$. Give a full deduction. Name all the logical axioms and inference rules involved in the deduction.

———————————— Solution:

1.	$\forall x[Px \to Qx]$	Σ_5
2.	$\forall x[Px \to Qx] \ \to \ [Pa \to Qa]$	(Q1)
3.	$Pa \to Qa$	1, 2, (PC)
4.	$\neg Qa$	Σ_5
5.	$\neg Pa$	3, 4, (PC)

Here is an another solution:

1.	$\forall x[Px \to Qx]$	Σ_5
2.	$\forall x[Px \to Qx] \ \to \ [Pa \to Qa]$	(Q1)
3.	$\neg Qa$	Σ_5
4.	$\neg Pa$	1, 2, 3, (PC)

———————————— End Solution

Let $\Sigma_6 = \{\forall x[Px \to Qx], Qa\}$.

(Continued on page 3.)

Problem 3

Give an $\mathcal{L}$-structure $\mathfrak{A}$ such that $\mathfrak{A} \models \Sigma_6$ and $\mathfrak{A} \not\models Pa$.

———————— Solution:

The universe A of the structure $\mathfrak{A}$ is given by $A = \{0\}$. Let $Q^{\mathfrak{A}} = A$, let $P^{\mathfrak{A}} = \emptyset$ and let $a^{\mathfrak{A}} = 0$.

Here is an another solution:

The universe A of the structure $\mathfrak{A}$ is the set of real numbers. Let $P^{\mathfrak{A}}$ be the set of all rational number; let $Q^{\mathfrak{A}} = A$ and let $a^{\mathfrak{A}}$ be the irrational number π (or any other real number that is not a rational number).

———————— End Solution

Problem 4

Let Σ be a set of $\mathcal{L}$-formulas and let ϕ an $\mathcal{L}$-formula. Explain why it follows from the Completeness Theorem that we either have

$$\text{there exists a } \Sigma\text{-deduction of } \phi \tag{1}$$

or

$$\text{there exists an } \mathcal{L}\text{-structure } \mathfrak{A} \text{ such that } \mathfrak{A} \models \Sigma \text{ and } \mathfrak{A} \not\models \phi \tag{2}$$

Explain why it follows from the Soundness Theorem that we do not have both (1) and (2).

———————— Solution:

Now, $\Sigma \models \phi$ holds, by definition, iff

$$\mathfrak{A} \models \Sigma \;\Rightarrow\; \mathfrak{A} \models \phi \quad \text{(any model for } \Sigma \text{ is also a model for } \phi)$$

Furthermore, $\Sigma \vdash \phi$ holds, by definition, iff

$$\text{there exists a } \Sigma\text{-deduction of } \phi\,.$$

The Completeness Theorem states that

$$\Sigma \models \phi \;\Rightarrow\; \Sigma \vdash \phi\,.$$

So why does it follow from the Completeness Theorem that we either have (1) or (2)?

Well, we have either $\Sigma \models \phi$ or $\Sigma \not\models \phi$ (this is obvious, ϕ will either follow – or not follow – logically from Σ). Assume that $\Sigma \models \phi$: By the Completeness Theorem, we have $\Sigma \vdash \phi$. By the definition of $\Sigma \vdash \phi$, we have (1). Assume that $\Sigma \not\models \phi$: By the definition of $\Sigma \models \phi$, we have (2). (It is not the case that every model for Σ is a model for ϕ. Thus, there has to be a structure $\mathfrak{A}$ which is a model for Σ but not for ϕ.)

The Soundness Theorem states that

$$\Sigma \vdash \phi \;\Rightarrow\; \Sigma \models \phi\,.$$

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Why does it follow from the Soundness Theorem that we cannot have both (1) and (2)?

Assume that (1) holds: By the definition above, we have $\Sigma \vdash \phi$. By the Soundness Theorem, we have $\Sigma \models \phi$. By the definition of $\Sigma \models \phi$, (2) does not hold. Assume that (2) holds: By the definition of $\Sigma \models \phi$, we have $\Sigma \not\models \phi$. By the Soundness Theorem we have $\Sigma \nvdash \phi$. By the definition of $\Sigma \vdash \phi$, (1) does not hold.

———————————— End Solution

Part II

Let $<$ be a binary relation symbol, let f be a binary function symbol, and let 0 and 1 be constant symbols. Let $\mathcal{L}$ be the language $\{<, f, 0, 1\}$. Let T^- be the $\mathcal{L}$-theory where we have the following non-logical axioms:

(T1) $\forall x[\neg x < x]$

(T2) $\forall xyz[x < y \wedge y < z \ \rightarrow \ x < z]$

(T3) $\forall xy[x < y \vee x = y \vee y < x]$

(T4) $\forall xy[x < y \ \rightarrow \ (x < f(x, y) \ \wedge \ f(x, y) < y)]$.

Let T be T^- extended with the axiom

(T5) $0 < 1$.

Problem 5

Prove that

$$T^- \vdash \forall xy[x < y \ \rightarrow \ \exists z[x < z \ \wedge \ z < y]]$$

by giving a T^--deduction. Name logical axioms and inference rules involved in the the deduction.

———————————— Solution:

Let $\top :\equiv \forall x[x = x] \vee \neg\forall x[x = x]$. The following deduction is very detailed. It is not necessary to give all the details (but it does not harm if you do).

(Continued on page 5.)

1.	$\forall xy[x < y \ \rightarrow \ (x < f(x,y) \ \wedge \ f(x,y) < y)]$	T^-
2.	$\forall xy[x < y \ \rightarrow \ (x < f(x,y) \ \wedge \ f(x,y) < y)] \ \rightarrow$ $\forall y[x < y \ \rightarrow \ (x < f(x,y) \ \wedge \ f(x,y) < y)]$	(Q1)
2′.	$\forall y[x < y \ \rightarrow \ (x < f(x,y) \ \wedge \ f(x,y) < y)] \ \rightarrow$ $x < y \ \rightarrow \ (x < f(x,y) \ \wedge \ f(x,y) < y)$	(Q1)
3.	$x < y \ \rightarrow \ (x < f(x,y) \ \wedge \ f(x,y) < y)$	1, 2, 2', (PC)
4.	$(x < f(x,y) \ \wedge \ f(x,y) < y) \ \rightarrow \ \exists z[x < z \ \wedge \ z < y]$	(Q2)
5.	$x < y \ \rightarrow \ \exists z[x < z \ \wedge \ z < y]$	3, 4, (PC)
6.	$\top \ \rightarrow \ (x < y \ \rightarrow \ \exists z[x < z \ \wedge \ z < y])$	5, (PC)
7.	$\top \ \rightarrow \ \forall y[x < y \ \rightarrow \ \exists z[x < z \ \wedge \ z < y]]$	6, (QR)
8.	$\top \ \rightarrow \ \forall xy[x < y \ \rightarrow \ \exists z[x < z \ \wedge \ z < y]]$	7, (QR)
9.	$\forall xy[x < y \ \rightarrow \ \exists z[x < z \ \wedge \ z < y]]$	8, (PC)

———————————— End Solution

Let t_0 denote the $\mathcal{L}$-term 0, and let t_{n+1} denote the $\mathcal{L}$-term $f(t_n, 1)$.

Problem 6

Prove that $T \vdash t_k < 1$ (for any natural number k).

———————————— Solution:

We prove $T \vdash t_k < 1$ by induction on k.

Let $k = 0$. We have $T \vdash 0 < 1$ by (T5) and, thus, $T \vdash t_0 < 1$ as t_0 is the term 0.

Assume by ind. hyp. that

$$T \vdash t_k < 1 \ . \tag{1}$$

By (T4), we have

$$T \vdash t_k < 1 \ \rightarrow \ (x < f(t_k, 1) \ \wedge \ f(t_k, 1) < 1) \ . \tag{2}$$

By (1), (2) and (PC), we have $T \vdash f(t_k, 1) < 1$. Thus, $T \vdash t_{k+1} < 1$ as t_{k+1} is the term $f(t_k, 1)$.

———————————— End Solution

Problem 7

Prove that $T^- \nvdash t_k < 1$ (for any natural number k).

———————————— Solution:

Let $\mathfrak{A}$ be the following $\mathcal{L}$-structure:

(Continued on page 6.)

- the universe is the set $\{a\}$ (so the universe contains only one element)
- $<^{\mathfrak{A}}$ is the empty set
- $f^{\mathfrak{A}}(a) = a$
- $0^{\mathfrak{A}} = 1^{\mathfrak{A}} = a$

Now, we have $\mathfrak{A} \models T^-$ (it is easy to check that $\mathfrak{A}$ satisfies all axioms of T^-, and $\mathfrak{A} \not\models t_k < 1$. By the Soundness Theorem, we have $T^- \nvdash t_k < 1$.

———————————— End Solution

Problem 8

Explain why any model for T is infinite.

———————————— Solution:

Let $\mathfrak{A} \models T$. We prove that $\mathfrak{A}$ is infinite.

Let the term t_k be defined as above, that is, let t_0 denote the $\mathcal{L}$-term 0, and let t_{n+1} denote the $\mathcal{L}$-term $f(t_n, 1)$.

We will prove that $t_i^{\mathfrak{A}} \neq t_j^{\mathfrak{A}}$ for any $i, j \in \mathbb{N}$ where $i \neq j$ (thus, there will be infinitely many elements in the universe of $\mathfrak{A}$).

By an induction argument similar to the one in the solution of Problem 6, we have $T \vdash t_k < t_{k+1}$ (for all $k \in \mathbb{N}$). By the Soundness Theorem, we have $\mathfrak{A} \models t_k < t_{k+1}$. Since $\mathfrak{A} \models (T2)$, we have $\mathfrak{A} \models t_j < t_k$ for any $i < k$. Since $\mathfrak{A} \models (T1)$, we have $t_i^{\mathfrak{A}} \neq t_k^{\mathfrak{A}}$ for any $i < k$. Thus, we also have $t_i^{\mathfrak{A}} \neq t_j^{\mathfrak{A}}$ for any $i, j \in \mathbb{N}$ such that $i \neq j$.

———————————— End Solution

Problem 9

Let $\mathfrak{A}$ and $\mathfrak{B}$ be two finite models for T^-. Prove that $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic.

———————————— Solution:

(Claim) Any finite model for T^- contains only one element.

First we prove that the claim holds.

Let $\mathfrak{A} \models T^-$, and let A denote the universe of $\mathfrak{A}$. Assume that we have $a, b \in A$ where $a \neq b$. Since $\mathfrak{A} \models (T3)$, we have $a <^{\mathfrak{A}} b$ or $b <^{\mathfrak{A}} a$. We can without loss of generality assume that $a <^{\mathfrak{A}} b$. Now, let $c_0 = a$ and let $c_{n+1} = f^{\mathfrak{A}}(c_n, b)$. We have $c_i \in A$ for all $i \in \mathbb{N}$. By an argument similar to the one in the solution of Problem 8, we have $c_i \neq c_j$ for any $i, j \in \mathbb{N}$ where $i \neq j$. (Use that $\mathfrak{A}$ satisfies (T1), (T2) and (T4).) This concludes the proof of the claim.

By the claim, it follows rather straightforwardly that any finite model for T^- is isomorphic to the model given in the solution of Problem 7.

———————————— End Solution

(Continued on page 7.)

Part III

Let $\mathfrak{N}$ be the standard $\mathcal{L}_{NT}$-structure, and let A be a set of $\mathcal{L}_{NT}$-axioms. We say that A *is ω-inconsistent* if there is a formula $\phi(x)$ such that $A \vdash \exists x[\phi(x)]$, but for each natural number n, $A \vdash \neg\phi(\overline{n})$. We say that A *is ω-consistent* if A is not ω-inconsistent.

Problem 10

Explain why $\mathfrak{N} \models A$ implies that A is ω-consistent.

Problem 11

Explain why A is ω-consistent implies that A is consistent.

END