Note: Lars Kristiansen, 4 pages. #1

Solution of Problem 8 and Problem 9.
The exam 2016, MAT-INF 3600

We prove (3) $\Rightarrow$ (2) $\Rightarrow$ (1) $\Rightarrow$ (3).

(3) $\Rightarrow$ (2).

$\mathfrak{M} \models \varphi$

$\Updownarrow$ (trivial)

$\mathfrak{M} \not\models \neg\varphi$

$\Downarrow$ (3)

$\Sigma \cup \{\neg\varphi\} \vdash \bot$

$\Updownarrow$ (PC and the Deduction Theorem)

$\Sigma \vdash \varphi$

(2) ⇒ (1).

Assume (I) does not hold.
Then there is $\mathfrak{A}$ and φ such that

$$\mathfrak{A} \models \Sigma \text{ and } \mathfrak{A} \models \neg\varphi \text{ and } \mathfrak{N} \models \varphi.$$

For this φ we have

$$\mathfrak{N} \models \varphi \text{ and } \Sigma \not\vdash \varphi.$$

Thus (2) does not hold.

by the Soundness Th

Hence (2) implies (I).

Note: Lars Kristiansen, 4 pages #3.

(1) ⇒ (3).

$\Sigma \cup \{\varphi\} \nvdash \bot$

⇕ (by definition)

$\Sigma \cup \{\varphi\}$ is consistent

⇕ (the Soundness and Completeness Theorem)

$\Sigma \cup \{\varphi\}$ has a model $\mathcal{A}$

⇕ (trivial)

$\mathcal{A} \models \Sigma$ and $\mathcal{A} \models \varphi$

⇓ (1)

$\Pi \models \varphi$

Note: Lars Kristiansen, 4pages #4

Problem 9.

(1) follows from (4)!

Why? Because

$$\mathfrak{A} \simeq \mathfrak{M} \Rightarrow \mathfrak{A} \equiv \mathfrak{M}$$

(Any structure isomorphic to $\mathfrak{M}$ is also elementarily equivalent to $\mathfrak{M}$.)

(4) does not follow (1)!

Why? Let Σ be the set of all true (in $\mathfrak{M}$) $\mathcal{L}_{NT}$-sentences. Then we have

$$\mathfrak{A} \models \Sigma \Rightarrow \mathfrak{A} \equiv \mathfrak{M}. \quad (1)$$

Let $\mathfrak{M}^*$ be the nonstandard structure known from Ch. 3 in L&K. We have

$$\mathfrak{M}^* \models \Sigma \text{ and } \mathfrak{M}^* \not\simeq \mathfrak{M}.$$

Thus (4) does not follow from (1).