UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT-INF3600 — Mathematical logic.

Day of examination: Monday, December 20, 2024.

Examination hours: 15:00 – 19:00.

This problem set consists of 9 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

The weights might be adjusted.

Feel free to refer to the Soundness Theorem, the Completeness Theorem and the Deduction Theorem.

PART I *(weight 40 %)*

Let P, Q, R, S be unary relation symbols and let $\mathcal{L}$ be the language $\{P, Q, R, S\}$. Let

$$\Sigma = \{\ \forall x[Px \wedge (Px \to (Qx \vee Rx))],\ \forall x[Qx \to Sx],\ \forall x[Rx \to Sx]\ \}\ .$$

Problem 1

Show that $\Sigma \vdash Sx$. Give a full Σ-deduction of Sx. Name all the axioms and inference rules involved in the deduction.

——————— Solution

1.	$\forall x[\,Px \wedge (Px \to (Qx \vee Rx))\,] \ \to\ Px \wedge (Px \to (Qx \vee Rx))$	(Q1)
2.	$\forall x[\,Px \wedge (Px \to (Qx \vee Rx))\,]$	Σ
3.	$Px \wedge (Px \to (Qx \vee Rx))$	1, 2, (PC)
4.	$\forall x[\,Qx \to Sx\,] \ \to\ Qx \to Sx$	(Q1)
5.	$\forall x[\,Qx \to Sx\,]$	Σ
6.	$Qx \to Sx$	4, 5, (PC)
7.	$\forall x[\,Rx \to Sx\,] \ \to\ Rx \to Sx$	(Q1)
8.	$\forall x[\,Rx \to Sx\,]$	Σ
9.	$Rx \to Sx$	7, 8, (PC)
10.	Sx	3, 6, 9, (PC)

(Continued on page 2.)

Problem 2

Prove that $\Sigma \nvdash \forall x[Qx \wedge Rx]$.

——————— Solution

Let $\mathfrak{A}$ be the $\mathcal{L}$-structure given by

- the universe A is the set $\{0\}$
- $P^{\mathfrak{A}} = Q^{\mathfrak{A}} = S^{\mathfrak{A}} = \{0\}$ and $R^{\mathfrak{A}} = \emptyset$.

We have $\mathfrak{A} \models \Sigma$ and $\mathfrak{A} \not\models \forall x[Qx \wedge Rx]$. By the Soundness Theorem, we have $\Sigma \nvdash \forall x[Qx \wedge Rx]$.

Problem 3

Is the set $\Sigma \cup \{\forall x[Qx \wedge Rx]\}$ consistent? Justify your answer.

——————— Solution

Let $\mathfrak{B}$ be the $\mathcal{L}$-structure given by

- the universe B is the set $\{0\}$
- $P^{\mathfrak{B}} = Q^{\mathfrak{B}} = S^{\mathfrak{B}} = \{0\}$ and $R^{\mathfrak{B}} = \{0\}$.

We have $\mathfrak{B} \models \Sigma \cup \{\forall x[Qx \wedge Rx]\}$. Thus, the set has a model. By the Soundness Theorem, the set is consistent.

Let $\Sigma' = \Sigma \cup \{ \exists x[\neg Sx] \}$.

Problem 4

Prove that $\Sigma' \vdash \bot$. Give a full Σ'-deduction of $\bot$. Name all the axioms and inference rules involved in the deduction.

——————— Solution

	$\vdots$	see Problem 1
10.	Sx	3, 6, 9, (PC)
11.	$\top \to \neg\neg Sx$	10, (PC)
12.	$\top \to \forall x[\neg\neg Sx]$	11, (QC)
13.	$\forall x[\neg\neg Sx]$	12, (PC)
14.	$\exists x[\neg Sx]$	Σ'
14.	$\neg\forall x[\neg\neg Sx]$	$:\equiv$, we rewrite 14
15.	$\bot$	13, 14, (PC)

(Continued on page 3.)

Problem 5

Let ϕ be any $\mathcal{L}$-sentence. Prove that at least one of the two sets

$$\Sigma \cup \{\phi\} \quad \text{and} \quad \Sigma \cup \{\neg\phi\}$$

is consistent.

——————— Solution

We know that Σ is consistent (see Problem 3). Assume for the sake of a contradiction that both sets are inconsistent, that is

$$\Sigma \cup \{\phi\} \vdash \bot \quad \text{and} \quad \Sigma \cup \{\neg\phi\} \vdash \bot \, .$$

By the Deduction Theorem, we have

$$\Sigma \vdash \phi \to \bot \quad \text{and} \quad \Sigma \vdash \neg\phi \to \bot \, .$$

Now, $\neg\phi$ follows tautologically from $\phi \to \bot$, and ϕ follows tautologically from $\neg\phi \to \bot$, and thus

$$\Sigma \vdash \neg\phi \quad \text{and} \quad \Sigma \vdash \phi \, .$$

This contradicts that Σ is consistent.

Problem 6

Prove that there exists a set of $\mathcal{L}$-sentences Γ such that (i) $\Sigma \subseteq \Gamma$, (ii) Γ is consistent and (iii) for any $\mathcal{L}$-sentence ϕ, we have $\phi \in \Gamma$ or $\neg\phi \in \Gamma$. Will the sentence $\forall x[Sx]$ necessarily be in Γ? Will the sentence $\forall x[Qx \wedge Rx]$ necessarily be in Γ? Will the sentence $\neg\forall x[Qx \wedge Rx]$ necessarily be in Γ? Answer the questions briefly, but justify your answers.

——————— Solution

The construction of the set Γ is standard. The construction is a part of the proof of the Completeness Theorem.

The sentence $\forall x[Sx]$ will for sure be in Γ as $\Sigma \cup \{\exists x[\neg Sx]\}$ is inconsistent (see problem 4). The sentence $\forall x[Qx \wedge Rx]$ might, or might not, be in Γ. It depends on the construction of Γ. The same goes for the sentence $\neg\forall x[Qx \wedge Rx]$. See promblem 3 and 4.

PART II *(weight 30 %)*

Comment: The problems in Part II have a wide range of acceptable solutions.

Let a and b be constant symbols, let f be a binary function symbol and let $\mathcal{L}$ be the language $\{a, b, f\}$. We define the *clean terms* inductively:

- a is a clean term
- ft_1t_2 is a clean term if t_1 and t_2 are clean terms.

The set of $\mathcal{L}$-formulas Σ is given by

$$\Sigma \; = \; \{ \; \neg t = b \; \mid t \text{ is a clean term } \} \, .$$

(Continued on page 4.)

Problem 7

Give a finite $\mathcal{L}$-structure $\mathfrak{A}$ such that $\mathfrak{A} \models \Sigma$. Give an infinite $\mathcal{L}$-structure $\mathfrak{B}$ such that $\mathfrak{B} \models \Sigma$.

———————————- Solution

We give the structure $\mathfrak{A}$: The universe A is the set $\{\ 0, 1\ \}$. Let $a^{\mathfrak{A}} = 0$ and $b^{\mathfrak{A}} = 1$. Let $f^{\mathfrak{A}}(0,0) = 0$, and let $f^{\mathfrak{A}}(0,1) = f^{\mathfrak{A}}(1,0) = f^{\mathfrak{A}}(1,1) = 1$. It is obvious that $\mathfrak{A} \models \Sigma$.

We give the structure $\mathfrak{B}$: The universe B is the set $\mathbb{N} \cup \{\omega\}$ (the set of natural numbers extended by an element ω, which is not a natural number). Let $a^{\mathfrak{B}} = 0$, $b^{\mathfrak{B}} = \omega$ and

$$f^{\mathfrak{B}}(x,y) \ = \ \begin{cases} x+1 & x \in \mathbb{N} \text{ and } y \in B \\ \omega & x = \omega \text{ and } y \in B. \end{cases}$$

For any clean term t, we have $t^{\mathfrak{B}} = n$ for some natural number n, whereas $b^{\mathfrak{B}} = \omega$. Thus, $\mathfrak{B}$ is an infinite model for Σ.

Problem 8

Give a set Δ_0 of $\mathcal{L}$-sentences such that (1) $\Sigma \cup \Delta_0$ is consistent and (2) any model for $\Sigma \cup \Delta_0$ is infinite. Prove that (1) and (2) holds.

———————————— Solution

Let $\Delta_0 = \{T_1, T_2\}$ where

$$T_1 :\equiv \forall xy[\neg a = f(x,y)] \quad \text{and} \quad T_2 :\equiv \forall x_1, x_2, y, z[f(x_1,y) = f(x_2,z) \to x_1 = x_2]\ .$$

The structure $\mathfrak{B}$ given above is a model for $\Sigma \cup \Delta_0$, and hence, $\Sigma \cup \Delta_0$ is consistent.

Next we prove that any model for $\Sigma \cup \Delta_0$ is infinite. We define a sequence of terms $s_0, s_1, s_2, \ldots$ inductively: $s_0 :\equiv a$ and $s_{n+1} :\equiv f(s_n, a)$.

(Claim) For any $n, m \in \mathbb{N}$, if $m \leq n$, then $\Delta_0 \vdash \neg s_m = s_{n+1}$.

We prove the claim by induction on n.

Assume $n = 0$. We have to prove that $\Delta_0 \vdash \neg s_0 = s_1$ where $s_0 = a$ and $s_1 = f(a,a)$. This holds by T_1.

Induction step: Let $m \leq n{+}1$. If $m = 0$, we have $\Delta_0 \vdash \neg s_0 = s_{n+2}$ by T_1. Let $1 \leq m \leq n{+}1$. By T_2, we have

$$\Delta_0 \vdash f(s_{m-1}, a) = f(s_{n+1}, a) \to s_{m-1} = s_{n+1}\ . \tag{i}$$

By our induction hypothesis we have

$$\Delta_0 \vdash \neg s_{m-1} = s_{n+1}\ . \tag{ii}$$

By (i) and (ii), we have

$$\Delta_0 \vdash \neg f(s_{m-1}, a) = f(s_{n+1}, a)\ .$$

This proves that the claim holds since $s_m :\equiv f(s_{m-1}, a)$ and $s_{n+2} :\equiv f(s_{n+1}, a)$.

It follows straightforwardly from the claim that any model for $\Sigma \cup \Delta_0$ is infinite.

(Continued on page 5.)

Problem 9

Give a *finite* set Δ_1 of $\mathcal{L}$-sentences such that

1. Δ_1 is consistent
2. any model for Δ_1 is infinite
3. for any formula ϕ, we have
$$\Sigma \vdash \phi \quad \Rightarrow \quad \Delta_1 \vdash \phi$$
(any formula that has a Σ-deduction will also have a Δ_1-deduction).

Prove that (1), (2) and (3) holds.

——————————— Solution

Let $\Delta_0 = \{T_1, T_2, T_3\}$ where

$$T_1 :\equiv \forall xy[\neg a = f(x,y)] \quad \text{and} \quad T_2 :\equiv \forall x_1, x_2, y, z[f(x_1,y) = f(x_2,z) \to x_1 = x_2] \quad \text{and} \quad T_3 :\equiv b = f(b,b)\ .$$

The structure $\mathfrak{B}$ given above is a model for Δ_1, and hence Δ_1 is consistent. Any model of Δ_1 is infinite since $\Delta_0 \subseteq \Delta_1$.

We prove

$$\text{for any clean term } t, \text{ we have } \Delta_1 \vdash \neg t = b \qquad \text{(claim)}$$

by induction on the structure of the clean term t.

If $t :\equiv a$, the claim holds by by T_1 and T_3. Let $t :\equiv ft_1t_2$. By T_2, we have

$$\Delta_1 \vdash f(t_1,t_2) = f(b,b) \to t_1 = b\ . \qquad (\texttt{I})$$

By our induction hypothesis we have

$$\Delta_1 \vdash \neg t_1 = b\ . \qquad (\texttt{II})$$

By (I) and (II), we have

$$\Delta_1 \vdash \neg f(t_1,t_2) = f(b,b)\ . \qquad (\texttt{III})$$

By (III) and T_3, we have $\Delta_1 \vdash \neg f(t_1,t_2) = b$, that is, $\Delta_1 \vdash \neg t = b$.

PART III *(weight 30 %)*

Let 0 be a constant symbol, let S be a unary function symbol, and let $<$ be a binary relation symbol. Let $\mathcal{L}_<$ be the language $\{0, S, <\}$. Let $N^<$ be the $\mathcal{L}_<$-theory consisting of the non-logical axioms

(A_1) $\forall x[\ \neg\, x < 0\ ]$

(A_2) $\forall xy[\ x < Sy \leftrightarrow (x < y \vee x = y)\ ]$.

(Continued on page 6.)

Problem 10

Prove that $N^< \vdash 0 < S0$. Prove that $N^< \cup \{0 = S0\} \vdash \bot$. (Sketch the deductions. There is no need to give the details.)

———————————- Solution

By using (A_2), we can deduce

$$0 < S0 \leftrightarrow (0 < 0 \vee 0 = 0)\ . \tag{1}$$

By using the logical axioms, we can deduce

$$0 = 0\ . \tag{2}$$

Now, $0 < S0$ follows from (1) and (2) by (PC). This proves that $N^< \vdash 0 < S0$.

By using (A_1), we can deduce $\neg 0 < 0$. Furthermore, by using logical axioms, we can deduce $\neg 0 < S0$ from $\neg 0 < 0$ and $0 = S0$. Hence, $N^< \cup \{0 = S0\} \vdash \neg 0 < S0$. We also have $N^< \cup \{0 = S0\} \vdash 0 < S0$ as $N^< \vdash 0 < S0$. This proves that $N^< \cup \{0 = S0\} \vdash \bot$.

We define the *basic formulas* inductively: an atomic $\mathcal{L}_<$-formula is a basic formula; $(\neg\alpha)$ is a basic formula if α is a basic formula; $(\alpha \vee \beta)$ is a basic formula if α and β are basic formulas.

A *basic sentence* is a basic formula with no free variables.

Problem 11

Prove that

$$N^< \vdash \phi \quad \text{or} \quad N^< \vdash \neg\phi$$

for any basic sentence ϕ.

——————————— Solution

Comment: This is a very hard problem, and that was taken into the account when we graded the exam.

We define the *numeral* $\overline{n}$ by $\overline{0} = 0$ and $\overline{n+1} = S\overline{n}$. Observe that any variable-free $\mathcal{L}_<$-term is a numeral. We need the following claim.

> **(Claim)** For any $a, b \in \mathbb{N}$, we have
>
> (A) $a = b \ \Rightarrow\ N^< \vdash \overline{a} = \overline{b}$
>
> (B) $a < b \ \Rightarrow\ N^< \vdash \overline{a} < \overline{b}$
>
> (C) $a \neq b \ \Rightarrow\ N^< \vdash \neg\overline{a} = \overline{b}$ and $N^< \vdash \neg\overline{b} = \overline{a}$
>
> (D) $a \not< b \ \Rightarrow\ N^< \vdash \neg\overline{a} < \overline{b}$.

We prove the claim by induction on $\max(a, b)$.

(Continued on page 7.)

Induction start. Assume $\max(a, b) = 1$. We need to prove

(a) $N^< \vdash \overline{0} = \overline{0}$ and $N^< \vdash \overline{1} = \overline{1}$

(b) $N^< \vdash \overline{0} < \overline{1}$

(c) $N^< \vdash \neg\overline{0} = \overline{1}$ and $N^< \vdash \neg\overline{1} = \overline{0}$

(d) $N^< \vdash \neg\overline{0} < \overline{0}$ and $N^< \vdash \neg\overline{1} < \overline{0}$ and $N^< \vdash \neg\overline{1} < \overline{1}$.

Clause (a) is obvious. Use (E1) and other logical axioms. Clause (b) is the first part of Problem 10.

We prove clause (c). By Problem 10, we have $N^< \cup \{\overline{0} = \overline{1}\} \vdash \bot$. By the Deduction Theorem, we have $N^< \vdash \overline{0} = \overline{1} \to \bot$. By (PC), we have $N^< \vdash \neg\overline{0} = \overline{1}$. Morover, by the logical axioms, we have $N^< \vdash \neg\overline{1} = \overline{0}$.

We prove clause (d). By (A_1), we have $N^< \vdash \neg\overline{0} < \overline{0}$ and $N^< \vdash \neg\overline{1} < \overline{0}$. At this stage we have proved that

$$N^< \vdash \neg\overline{1} < \overline{0} \quad \text{and} \quad N^< \vdash \neg\overline{1} = \overline{0} \tag{3}$$

By (3) and (PC), we have

$$N^< \vdash \neg(\overline{1} < \overline{0} \vee \overline{1} = \overline{0}) \tag{4}$$

By (A_2), we have

$$\overline{1} < \overline{1} \to (\overline{1} < \overline{0} \vee \overline{1} = \overline{0}) \tag{5}$$

By (4), (5) and (PC), we have $N^< \vdash \neg\overline{1} < \overline{1}$.

Induction step. Assume that the claim holds when $k = \max(a, b) \geq 1$, that is, our induction hypothesis will be that the claim holds for any $a, b \leq k$. We prove that the claim holds for for any $a, b \leq k + 1$.

The proof of clause (A) is trivial. Use (E2).

We prove clause (B). We will argue that we have

$$N^< \vdash \overline{a} < \overline{k+1} \tag{6}$$

for any $a < k+1$. Pick an arbitrary a strictly less than $k+1$. Our proof splits into the case $a = k$ and $a < k$. In the case when $a = k$, we have $N^< \vdash \overline{a} = \overline{k}$ by clause (A), and then by the right-left direction of (A_2), we have $N^< \vdash \overline{a} < \overline{k+1}$. In the case when $a < k$, we have $N^< \vdash \neg\overline{a} < \overline{k}$ by the induction hypothesis, and then, again by the right-left direction of (A_2), we have $N^< \vdash \overline{a} < \overline{k+1}$. This proves (6). It follows rather straightforwardly from (6) that (B) holds for any $a, b \leq k + 1$.

We prove clause (C). We will prove

$$N^< \vdash \neg\overline{a} = \overline{k+1} \tag{7}$$

for any $a < k + 1$. It is well-know that (7) is equivalent to

$$N^< \cup \{\overline{a} = \overline{k+1}\} \vdash \bot\,. \tag{8}$$

(Continued on page 8.)

In order to prove (8), pick an arbitrary a strictly less than $k+1$. By clause (B) (which we already have proved) we have

$$N^< \vdash \overline{a} < \overline{k+1}\,. \tag{9}$$

Thus,

$$N^< \cup \{\overline{a} = \overline{k+1}\} \vdash \overline{a} < \overline{a}\,. \tag{10}$$

By our induction hypothesis, we have

$$N^< \vdash \neg\overline{a} < \overline{a}\,. \tag{11}$$

It follows from (10) and (11) that (8) holds, and then, (7) also holds. This completes the proof of (7). It follows rather straightforwardly from (7) that (C) holds for any $a, b \leq k+1$ (recall that we can deduce $\neg s = t$ from $\neg t = s$ by using the logical axioms).

We prove (D). We will prove by a secondary induction on b that we have

$$N^< \vdash \neg\overline{k+1} < \overline{b} \tag{12}$$

for any $b \leq k+1$. Let $b = 0$. Then (12) holds by (A_1). Let $b = c+1$. By our induction hypothesis on b, we have

$$N^< \vdash \neg\overline{k+1} < \overline{c}\,. \tag{13}$$

Now, $k+1 \neq c$ as $c+1 = b \leq k+1$. Thus, by (C) (which we already have proved) we have

$$N^< \vdash \neg\overline{k+1} = \overline{c}\,. \tag{14}$$

By (13), (14) and (PC), we have

$$N^< \vdash \neg(\overline{k+1} < \overline{c} \vee \overline{k+1} = \overline{c})\,. \tag{15}$$

It follows from (15) and the left-right direction of (A_2) that (12) holds. This completes our proof of (12).

It follows rather straightforwardly from (12) that (D) holds for any $a, b \leq k+1$.

Now we have proved the claim. Next we prove by induction on the structure of a basic sentence ϕ that we have

$$N^< \vdash \phi \quad \text{or} \quad N^< \vdash \neg\phi\,. \tag{$\dagger$}$$

The proof splits into the cases: (i) ϕ is atomic; (ii) $\phi :\equiv (\neg\alpha)$; (iii) $\phi :\equiv (\alpha \vee \beta)$.

(i) In this case (†) follows straightforwardly from **(Claim)**.

(ii) By our induction hypothesis we have

$$N^< \vdash \alpha \quad \text{or} \quad N^< \vdash \neg\alpha\,.$$

We can deduce $(\neg(\neg\alpha))$ from α by using the inference rule (PC), and thus, (†) holds.

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(iii) By our induction hypothesis we have

$$N^{<} \vdash \alpha \quad \text{or} \quad N^{<} \vdash \neg\alpha$$

and

$$N^{<} \vdash \beta \quad \text{or} \quad N^{<} \vdash \neg\beta \, .$$

The proof splits into the subcases: (iiia) $N^{<} \vdash \alpha$ or $N^{<} \vdash \beta$ and (iiib) it is not the case that $N^{<} \vdash \alpha$ or $N^{<} \vdash \beta$. In case (iiia), we have $N^{<} \vdash \alpha \vee \beta$; In case (iiib), we have $N^{<} \vdash \neg(\alpha \vee \beta)$. We skip the details.

We define the *bounded formulas* inductively: an atomic $\mathcal{L}_{<}$-formula is a bounded formula; $(\neg\alpha)$ is a bounded formula if α is a bounded formula; $(\alpha \vee \beta)$ is a bounded formula if α and β are bounded formulas; $\forall x[\, x < t \to \alpha \,]$ is a bounded formula if α is a bounded formula, x is a variable and t is an $\mathcal{L}_{<}$-term with no occurrences of x.

A *bounded sentence* is a bounded formula with no free variables.

Problem 12

Explain why we have

$$N^{<} \vdash \phi \quad \text{or} \quad N^{<} \vdash \neg\phi$$

for any bounded sentence ϕ. Try to sketch a proof.

——————————— Solution

This can be proved by induction over the structure of ϕ. To deal with the case when ϕ is of the form $\phi :\equiv \forall x[\, x < t \to \alpha \,]$, wee will need a variant of Rosser's Lemma (see page 125 in Leary & Kristiansen), eg., the variant

$$N^{<} \vdash \forall x \left[\, x < S\overline{n} \;\; \leftrightarrow \;\; (x = \overline{0} \vee x = \overline{1} \vee \ldots \vee x = \overline{n}) \,\right] \, .$$

END