UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT-INF3600 — Mathematical logic.

Day of examination: Monday, December 8, 2023.

Examination hours: 15:00 – 19:00.

This problem set consists of 7 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

The weights might be adjusted.

PART I *(weight 40 %)*

Let P and Q be unary relation symbols, let c be a constant symbol and let $\mathcal{L}$ be the language $\{c, P, Q\}$. Furthermore, let

$$\Sigma = \{\, Qc\,,\ \neg Pc\,,\ \exists x[\,\neg Qx\,]\,,\ \exists x[\,Px\,]\,\}\,.$$

Problem 1 *(weight 5 %)*

State the Soundness Theorem for first-order logic.

——————————— Solution:

Let Σ be a set of $\mathcal{L}$-formulas. We state two equivalent versions of the Soundness Theorem.

Version I: We have

$$\Sigma \vdash \phi \quad \Rightarrow \quad \Sigma \models \phi$$

for any $\mathcal{L}$-formula ϕ.

Version II: If Σ has a model, then Σ is consistent.

Problem 2 *(weight 5 %)*

Give an $\mathcal{L}$-structure $\mathfrak{A}$ such that $\mathfrak{A} \models \Sigma$ and $\mathfrak{A} \models \forall x[Px \vee Qx]$.

——————————— Solution:

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(Continued on page 2.)

One possible answer: Let the universe A be the set $\{0, 1\}$. Furthermore, let $Q^{\mathfrak{A}} = \{0\}$; let $P^{\mathfrak{A}} = \{1\}$; let $c^{\mathfrak{A}} = 0$.

Another possible answer: Let the universe A be the set of natural numbers. Furthermore, let $Q^{\mathfrak{A}}$ be the set of odd numbers; $P^{\mathfrak{A}}$ be the set of even numbers; let $c^{\mathfrak{A}} = 1729$.

Problem 3 *(weight 5 %)*

Give an $\mathcal{L}$-structure $\mathfrak{B}$ such that $\mathfrak{B} \models \Sigma$ and $\mathfrak{B} \not\models \forall x[Px \vee Qx]$.

——— Solution:

Let the universe B be the set $\{0, 1, 2\}$. Furthermore, let $Q^{\mathfrak{B}} = \{0\}$; let $P^{\mathfrak{A}} = \{1\}$; let $c^{\mathfrak{B}} = 1$.

Problem 4 *(weight 5 %)*

Do we have $\Sigma \vdash \forall x[Px \vee Qx]$? Do we have $\Sigma \vdash \neg\forall x[Px \vee Qx]$? Justify your answers.

——— Solution:

We do not have $\Sigma \vdash \forall x[Px \vee Qx]$. By Problem 3, we have a structure $\mathfrak{B}$ such that $\mathfrak{B} \models \Sigma$ and $\mathfrak{B} \not\models \forall x[Px \vee Qx]$. Thus, $\Sigma \not\models \forall x[Px \vee Qx]$. Thus, by the Soundness Theorem (Version I, Problem 1), we have $\Sigma \nvdash \forall x[Px \vee Qx]$.

We do not have $\Sigma \vdash \neg\forall x[Px \vee Qx]$. By Problem 2, we have a structure $\mathfrak{A}$ such that $\mathfrak{A} \models \Sigma$ and $\mathfrak{A} \not\models \neg\forall x[Px \vee Qx]$. Thus, $\Sigma \not\models \neg\forall x[Px \vee Qx]$. Thus, by the Soundness Theorem (Version I, Problem 1), we have $\Sigma \nvdash \neg\forall x[Px \vee Qx]$.

Problem 5 *(weight 5 %)*

What does it mean that a set of $\mathcal{L}$-formulas is consistent? Give the definition.

——— Solution:

A set of $\mathcal{L}$-formulas is consistent (by definition) if $\Sigma \nvdash \bot$ (where $\bot :\equiv \forall x[x = x] \wedge \neg\forall x[x = x]$).

Problem 6 *(weight 5 %)*

Is the set $\Sigma \cup \{\forall x[Px \vee Qx]\}$ consistent? Is the set $\Sigma \cup \{\neg\forall x[Px \vee Qx]\}$ consistent? Justify your answers.

——— Solution:

By Problem 2, the set $\Sigma \cup \{\forall x[Px \vee Qx]\}$ has a model. By Problem 1 (the Soundness Theorem, Version II), $\Sigma \cup \{\forall x[Px \vee Qx]\}$ is a consistent set.

By Problem 3, the set $\Sigma \cup \{\neg\forall x[Px \vee Qx]\}$ has a model. By Problem 1 (the Soundness Theorem, Version II), $\Sigma \cup \{\neg\forall x[Px \vee Qx]\}$ is a consistent set.

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Problem 7 *(weight 5 %)*

State the Completeness Theorem for first-order logic. State the Deduction Theorem for first-order logic.

——————————————— Solution:

Let Σ be a set of $\mathcal{L}$-formulas.

The Completeness Theorem: We have

$$\Sigma \models \phi \quad \Rightarrow \quad \Sigma \vdash \phi$$

for any $\mathcal{L}$-formula ϕ.

The Deduction Theorem: We have

$$\Sigma, \eta \vdash \xi \quad \Leftrightarrow \quad \Sigma \vdash \eta \to \xi$$

for any $\mathcal{L}$-sentence η.

Problem 8 *(weight 5 %)*

Let $\mathcal{L}$ be any first-order language and let Σ be a set of $\mathcal{L}$-sentences. Furthermore, let ψ be an $\mathcal{L}$-formula such that $\Sigma \models \psi$. Explain why there exists an $\mathcal{L}$-sentence ϕ such that

$$\vdash \phi \to \psi \quad \text{and} \quad \Sigma \vdash \phi \, .$$

——————————————— Solution:

Assume Σ is a set of sentences. Assume $\Sigma \models \psi$. By the Completeness Theorem, we have $\Sigma \vdash \psi$. By the definition of $\vdash$, we have a Σ-deduction $\psi_1, \ldots \psi_n$ of ψ. Let

$$\Sigma_0 \;=\; \Sigma \,\cup\, \{\, \psi_1, \ldots \psi_n \,\} \, .$$

The Σ_0 is a finite subset of Σ, and moreover, $\Sigma_0 \vdash \psi$.

Let $\Sigma_0 = \{\phi_1, \ldots, \phi_k\}$. Every formula in Σ_0 is a sentence. By repeated application of the Deduction Theorem, we have

$$\vdash \phi_1 \wedge \ldots \wedge \phi_k \to \psi \, .$$

It is obvious that $\Sigma \vdash \phi_1 \wedge \ldots \wedge \phi_k$. Thus, let $\phi :\equiv \phi_1 \wedge \ldots \wedge \phi_k$, and we have $\vdash \phi \to \psi$ and $\Sigma \vdash \phi$.

PART II *(weight 35 %)*

Problem 9 *(weight 5 %)*

State the axiom scheme (E3).

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——— Solution:

We have the axiom

$$x_1 = y_1 \wedge \ldots \wedge x_n = y_n \ \rightarrow \ (\ Rx_1 \ldots x_n \ \rightarrow \ Ry_1 \ldots y_n \ ) \tag{E3}$$

for any $n > 0$, any $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathit{Vars}$ and any n-ary relation symbol R.

Problem 10 *(weight 10 %)*

Let R be a unary relation symbol, and let x and y be variables. Prove that

$$\vdash Rx \rightarrow (x = y \rightarrow Ry)$$

Give a detailed deduction of the formula. Name alle the logical axioms and inference rules involved in the deduction.

——— Solution:

1.	$x = y \rightarrow (Rx \rightarrow Ry)$	(E3)
2.	$Rx \rightarrow (x = y \rightarrow Ry)$	1, (PC)

Problem 11 *(weight 10 %)*

Let R be a unary relation symbol, and let $x_1, \ldots, x_n$ and y be variables. Prove that

$$\vdash (\ Rx_1 \wedge \ldots \wedge Rx_n \ ) \rightarrow (\ (\ x_1 = y \vee \ldots \vee x_n = y \ ) \rightarrow Ry \ )$$

Use induction on n.

——— Solution:

We prove

$$\vdash (\ Rx_1 \wedge \ldots \wedge Rx_n \ ) \rightarrow (\ (\ x_1 = y \vee \ldots \vee x_n = y \ ) \rightarrow Ry \ ) \tag{†}$$

by induction on n.
When $n = 1$, (†) holds by Problem 10.
Induction step: Assume (*). Then we have a deduction of

$$(\ Rx_1 \wedge \ldots \wedge Rx_n \ ) \rightarrow (\ (\ x_1 = y \vee \ldots \vee x_n = y \ ) \rightarrow Ry \ ) \tag{*}$$

By Problem 10, we have a deduction of

$$Rx_{n+1} \rightarrow (x_{n+1} = y \rightarrow Ry) \tag{**}$$

Now,

$$(\ Rx_1 \wedge \ldots \wedge Rx_n \wedge Rx_{n+1} \ ) \rightarrow (\ (\ x_1 = y \vee \ldots \vee x_n = y \wedge x_{n+1} = y \ ) \rightarrow Ry \ )$$

follows from (*) and (**) by one application of (PC).
Thus, we conclude that (†) holds for all $n > 0$.

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Problem 12 *(weight 10 %)*

Let R be a unary relation symbol, and let $c_1, \ldots, c_n$ be constant symbols. Furthermore, let $\Sigma = \{\, \forall x[x = c_1 \vee \ldots \vee x = c_n] \,\}$. Prove that

$$\Sigma \vdash (\, Rc_1 \wedge \ldots \wedge Rc_n \,) \;\leftrightarrow\; \forall y[\, Ry \,] \,.$$

Sketch a Σ-deduction of the formula $(Rc_1 \wedge \ldots \wedge Rc_n) \leftrightarrow \forall y[Ry]$.

——————————————— Solution:

We sketch a Σ-deduction of

$$(Rc_1 \wedge \ldots \wedge Rc_n) \rightarrow \forall y[Ry] \tag{1}$$

and Σ-deduction of

$$\forall y[Ry] \rightarrow (Rc_1 \wedge \ldots \wedge Rc_n) \,. \tag{2}$$

The formula $(Rc_1 \wedge \ldots \wedge Rc_n) \leftrightarrow \forall y[Ry]$ follows from (1) and (2) by one application of (PC).

The Deduction of (1): By Problem 11, we can deduce

$$(\; Rx_1 \wedge \ldots \wedge Rx_n \;) \rightarrow (\; (\; x_1 = y \vee \ldots \vee x_n = y \;) \rightarrow Ry \;) \tag{3}$$

by using logical axioms. From (3) we can deduce

$$(\; Rc_1 \wedge \ldots \wedge Rc_n \;) \rightarrow (\; (\; c_1 = y \vee \ldots \vee c_n = y \;) \rightarrow Ry \;) \tag{4}$$

by using logical axioms. Since $\forall x[x = c_1 \vee \ldots \vee x = c_n]$ is in Σ, we can find a Σ-deduction of

$$c_1 = y \vee \ldots \vee c_n = y \,. \tag{5}$$

Now

$$(\; Rc_1 \wedge \ldots \wedge Rc_n \;) \rightarrow Ry \tag{6}$$

follows from (4) and (5) by one application of (PC), and

$$(\; Rc_1 \wedge \ldots \wedge Rc_n \;) \rightarrow \forall y[Ry]$$

follows from (6) by one application of (QR).

The Deduction of (2): This is a pure logical deduction. We need only logical axioms. For $i = 1, \ldots, n$, use the instance

$$\forall y[Ry] \rightarrow Rc_i$$

of the axiom scheme (Q1). Then use (PC) to deduce (2).

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PART III *(weight 25 %)*

Let $\mathcal{L}^-_{NT}$ be the language $\{0, S, +, \cdot\}$. We define the *numeral* $\overline{n}$ by $\overline{0} = 0$ and $\overline{n+1} = S\overline{n}$ (this is the standard definition of the numerals). Let N^- be the $\mathcal{L}^-_{NT}$-theory consisting of the non-logical axioms

(A_1) $\forall x[x = \overline{0} \vee x = \overline{1}]$

(A_2) $S\overline{1} = \overline{0}$

(A_3) $\forall x[\ x + 0 = x\ ]$

(A_4) $\forall xy[\ x + Sy = S(x + y)\ ]$

(A_5) $\forall x[\ x \cdot 0 = 0\ ]$

(A_6) $\forall x[\ x \cdot Sy = (x \cdot y) + x\ ]$.

Problem 13 *(weight 10 %)*

Prove that we have

$$N^- \vdash t = \overline{0} \quad \text{or} \quad N^- \vdash t = \overline{1}$$

for any variable-free $\mathcal{L}^-_{NT}$-term t.

——————————————— Solution:

We prove

$$N^- \vdash t = \overline{0} \quad \text{or} \quad N^- \vdash t = \overline{1} \qquad (*)$$

by induction over the structure of the variable-free term t. The cases will be $t :\equiv 0$; $t :\equiv St_1$; $t :\equiv t_1 + t_2$; $t :\equiv t_1 \cdot t_2$. We will reason in an arbitrary model for N^-.

Case $t :\equiv 0$. We have $N^- \vdash t = \overline{0}$ by (E1) and other logical axioms.

Case $t :\equiv St_1$. By ind. hyp., we have (i) $N^- \vdash t_1 = \overline{0}$ or (ii) $N^- \vdash t_1 = \overline{1}$. Case (i): We have $N^- \vdash t_1 = \overline{0}$. By (E2) and other logical axioms, we have $N^- \vdash t = \overline{1}$ Case (ii): By (A_2) and logical axioms, we have $N^- \vdash t = \overline{0}$.

Case $t :\equiv t_1 + t_2$. By ind. hyp., we can split the proof into the cases

- (i) $N^- \vdash t_1 = 0$ and $N^- \vdash t_2 = 0$
- (ii) $N^- \vdash t_1 = 0$ and $N^- \vdash t_2 = S0$
- (iii) $N^- \vdash t_1 = S0$ and $N^- \vdash t_2 = 0$
- (iv) $N^- \vdash t_1 = S0$ and $N^- \vdash t_2 = S0$.

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Case (i). In an arbitrary model for N^-, we have

$$t_1 + t_2 \;=\; 0 + 0 \;\overset{(A_3)}{=}\; 0\,.$$

Thus, by the Completeness Theorem, $N^- \vdash t = \overline{0}$.

Case (ii). In an arbitrary model for N^-, we have

$$t_1 + t_2 \;=\; 0 + S0 \;\overset{(A_4)}{=}\; S(0+0) \;\overset{(A_3)}{=}\; S(0)\,.$$

Thus, by the Completeness Theorem, $N^- \vdash t = \overline{1}$.

Case (iii). In an arbitrary model for N^-, we have

$$t_1 + t_2 \;=\; S0 + 0 \;\overset{(A_3)}{=}\; S(0)\,.$$

Thus, by the Completeness Theorem, $N^- \vdash t = \overline{1}$.

Case (ii). In an arbitrary model for N^-, we have

$$t_1 + t_2 \;=\; S0 + S0 \;\overset{(A_4)}{=}\; S(S0+0) \;\overset{(A_3)}{=}\; SS(0) \;\overset{(A_2)}{=}\; 0\,.$$

Thus, by the Completeness Theorem, $N^- \vdash t = \overline{0}$.

Case $t :\equiv t_1 \cdot t_2$. This case is similar to the preceding case.

Problem 14 *(weight 15 %)*

How many models has N^- up to isomorphism? (Give a number k such that any model for N^- will be isomorphic to one of these k models.)

Can you say something about how many models N^- has up to elementary equivalence?

Extend N^- to a complete theory, and prove that your extension is complete.

——————————————— Solution:

The number k is 2 (this should be justified), and N^- has also 2 models up to elementary equivalence. If we extend N^- withe the axiom $\neg\overline{0} = \overline{1}$ (or with the axiom $\overline{0} = \overline{1}$), then N^- has one model up to isomorphism. Let $\mathfrak{A}$ that unique model. If a sentence ϕ is true-in-$\mathfrak{A}$, the ϕ is true in all models for N^-, that is

$$\mathfrak{A} \models \phi \;\;\Rightarrow\;\; N^- \models \phi\,.$$

By the Completeness Theorem, we have

$$\mathfrak{A} \models \phi \;\;\Rightarrow\;\; N^- \models \phi \;\;\Rightarrow\;\; N^- \vdash \phi\,.$$

For any sentence $\mathcal{L}^-_{NT}$-sentece θ, we have either $\mathfrak{A} \models \theta$ or $\mathfrak{A} \models \neg\theta$. Thus, we also have either $N^- \vdash \theta$ or $N^- \vdash \neg\theta$. This shows that our extension of N^- is a complete theory (the theory is consistent since it has a model).

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