UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in	MAT-INF 4130 — Numerical linear algebra
Day of examination:	3 December 2013
Examination hours:	1100 – 1500
This problem set consists of 4 pages.	
Appendices:	None
Permitted aids:	None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All 9 part questions will be weighted equally.

Problem 1 True or false

Give reasons for your answers.

1a

If two matrices have the same eigenvalues they must be similar.

Answer: False. The matrices $\boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\boldsymbol{J} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ have the same eigenvalues $\lambda_1 = \lambda_2 = 1$. However $\boldsymbol{J}$ is defective and cannot be diagonalized by a similarity transformation.

1b

If $\boldsymbol{x} \in \operatorname{span}(\boldsymbol{A})$ and $\boldsymbol{y} \in \ker(\boldsymbol{A})$ then $\boldsymbol{x}^T\boldsymbol{y} = 0$ for any $\boldsymbol{A} \in \mathbb{R}^{2\times 2}$.

Answer: False. Let $\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Then $\boldsymbol{x} := \boldsymbol{A}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \in \operatorname{span}(\boldsymbol{A})$ and $\boldsymbol{y} := \begin{bmatrix} 1 \\ -1 \end{bmatrix} \in \ker(\boldsymbol{A})$ since $\boldsymbol{A}\boldsymbol{y} = \boldsymbol{0}$. But $\boldsymbol{x}^T\boldsymbol{y} = 2 \neq 0$.

1c

The overdetermined linear system

$$\boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix} = \boldsymbol{b}$$

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has a least squares solution $x_1 = -6$, $x_2 = 9/2$. This solution is unique.

Answer: Both true. The normal equation $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^T\boldsymbol{b}$ for this system is

$$\begin{bmatrix}1 & 2 & 4\\ 2 & 3 & 5\end{bmatrix}\begin{bmatrix}1 & 2\\ 2 & 3\\ 4 & 5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}21 & 28\\ 28 & 38\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}1 & 2 & 4\\ 2 & 3 & 5\end{bmatrix}\begin{bmatrix}2\\ 3\\ -2\end{bmatrix} = \begin{bmatrix}0\\ 3\end{bmatrix}$$

with solution $x_1 = -6$, $x_2 = 9/2$. The solution is unique since $\boldsymbol{A}^T\boldsymbol{A}$ is nonsingular. Uniqueness also follows since $\boldsymbol{A}$ has linearly independent columns.

1d

The matrix

$$\boldsymbol{A} := \begin{bmatrix}1 & -1 & 0 & 1\\ 0 & 1 & -1 & -1\\ 1 & 0 & 5 & -10\\ 0 & 9 & 0 & 10\end{bmatrix}$$

has a unique LU-factorization. (Do not compute the factorization.)

Answer: True. $\boldsymbol{A}$ has a unique LU-factorization if and only if the leading principal submatrices $\boldsymbol{A}_{[k]}$ are nonsingular for $k = 1, 2, 3$. $\boldsymbol{A}_{[1]} = [1]$ is nonsingular since it is nonzero, and $\boldsymbol{A}_{[2]}$ is triangular with nonzero diagonal elements and therefore nonsingular. We show that $\boldsymbol{A}_{[3]}$ has linearly independent columns and is therefore nosingular. We find

$$\boldsymbol{A}_{[3]}\boldsymbol{x} = \boldsymbol{0} \iff \begin{matrix}x_1 - x_2 = 0\\ x_2 - x_3 = 0\\ x_1 + 5x_3 = 0\end{matrix} \iff x_1 = x_2 = x_3 = 0.$$

Alternatively,

$$\det(\boldsymbol{A}_{[3]}) = \det(\begin{bmatrix}1 & -1\\ 0 & 5\end{bmatrix}) + \det(\begin{bmatrix}-1 & 0\\ 1 & -1\end{bmatrix}) = 5 + 1 = 6 \neq 0.$$

Problem 2 Givens rotation

A Givens rotation of order 2 has the form $\boldsymbol{G} := \begin{bmatrix}c & s\\ -s & c\end{bmatrix} \in \mathbb{R}^{2\times 2}$, where $s^2 + c^2 = 1$.

2a

Is $\boldsymbol{G}$ symmetric and unitary?

Answer: $\boldsymbol{G}$ is only symmetric for $s = 0$. $\boldsymbol{G}$ is unitary since $\boldsymbol{G}^T\boldsymbol{G} = \begin{bmatrix}s^2 + c^2 & 0\\ 0 & s^2 + c^2\end{bmatrix} = \boldsymbol{I}$.

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2b

Given $x_1, x_2 \in \mathbb{R}$ and set $r := \sqrt{x_1^2 + x_2^2}$. Find $\boldsymbol{G}$ and y_1, y_2 so that $y_1 = y_2$, where $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \boldsymbol{G} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Answer: We find $y_1 = y_2$ if and only if $cx_1 + sx_2 = -sx_1 + cx_2$ and $s^2 + c^2 = 1$. Thus s and c must be solutions of

$$\begin{aligned} (x_1 + x_2)s + (x_1 - x_2)c &= 0 \\ s^2 + c^2 &= 1. \end{aligned}$$

If $x_1 = x_2$ then the solution is $s = 0$ and $c = \pm 1$. Suppose $x_1 \neq x_2$. Substituting $c = \frac{x_1 + x_2}{x_2 - x_1} s$ into $1 = s^2 + c^2$ we find

$$1 = s^2\big(1 + \frac{(x_1 + x_2)^2}{(x_2 - x_1)^2}\big) = s^2 \frac{(x_2 - x_1)^2 + (x_1 + x_2)^2}{(x_2 - x_1)^2} = s^2 \frac{2r^2}{(x_2 - x_1)^2}.$$

There are two solutions

$$s_1 = \frac{x_2 - x_1}{r\sqrt{2}}, \; c_1 = \frac{x_2 + x_1}{r\sqrt{2}}, \quad s_2 = -s_1, \; c_2 = -c_1.$$

We find

$$y_1 = y_2 = c_1 x_1 + s_1 x_2 = \frac{1}{r\sqrt{2}}\big((x_1 + x_2)x_1 + (x_2 - x_1)x_2\big) = r/\sqrt{2}.$$

The other solution is $y_1 = y_2 = c_2 x_1 + s_2 x_2 = -(c_1 x_1 + s_1 x_2) = -r/\sqrt{2}$.

Problem 3 Perturbation of the identity matrix

Let $\boldsymbol{B} \in \mathbb{R}^{n \times n}$ and suppose $\|\boldsymbol{B}\| < 1$ for some operator norm.

3a

Show that $\boldsymbol{I} - \boldsymbol{B}$ is nonsingular.

Answer: Suppose $\boldsymbol{I} - \boldsymbol{B}$ is singular. Then $(\boldsymbol{I} - \boldsymbol{B})\boldsymbol{x} = \boldsymbol{0}$ for some nonzero $\boldsymbol{x} \in \mathbb{C}^n$, and $\boldsymbol{x} = \boldsymbol{B}\boldsymbol{x}$ so that $\|\boldsymbol{x}\| = \|\boldsymbol{B}\boldsymbol{x}\| \le \|\boldsymbol{B}\|\|\boldsymbol{x}\|$. But then $\|\boldsymbol{B}\| \ge 1$. It follows that $\boldsymbol{I} - \boldsymbol{B}$ is nonsingular if $\|\boldsymbol{B}\| < 1$.

3b

Show that

$$\|(\boldsymbol{I} - \boldsymbol{B})^{-1}\| \le \frac{1}{1 - \|\boldsymbol{B}\|}.$$

Answer: Taking norms and using the inverse triangle inequality in

$$\boldsymbol{I} = (\boldsymbol{I} - \boldsymbol{B})(\boldsymbol{I} - \boldsymbol{B})^{-1} = (\boldsymbol{I} - \boldsymbol{B})^{-1} - \boldsymbol{B}(\boldsymbol{I} - \boldsymbol{B})^{-1}$$

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implies

$$\|\boldsymbol{I}\| \geq \|(\boldsymbol{I}-\boldsymbol{B})^{-1}\| - \|\boldsymbol{B}(\boldsymbol{I}-\boldsymbol{B})^{-1}\| \geq \big(1-\|\boldsymbol{B}\|\big)\|(\boldsymbol{I}-\boldsymbol{B})^{-1}\|.$$

If the matrix norm is an operator norm then $\|\boldsymbol{I}\| = 1$ and the upper bound follows.

Problem 4 Matlab program

Suppose $\boldsymbol{A} \in \mathbb{R}^{m\times n}, \boldsymbol{b} \in \mathbb{R}^m$, where $\boldsymbol{A}$ has rank n and let $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$ be a singular value factorization of $\boldsymbol{A}$. Thus $\boldsymbol{U} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{\Sigma}, \boldsymbol{V} \in \mathbb{R}^{n\times n}$. Write a Matlab `function [x,K]=lsq(A,b)` that uses the singular value factorization of $\boldsymbol{A}$ to calculate a least squares solution $\boldsymbol{x} = \boldsymbol{V}\boldsymbol{\Sigma}^{-1}\boldsymbol{U}^T\boldsymbol{b}$ to the system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ and the spectral (2-norm) condition number of $\boldsymbol{A}$. The Matlab command `[U,Sigma,V]=svd(A,0)` computes the singular value factorization of $\boldsymbol{A}$.

Answer: The matrix $\boldsymbol{\Sigma}$ is a diagonal matrix with the singular values on the diagonal ordered so that $\sigma_1 \geq \cdots \geq \sigma_n$. Moreover, $\sigma_n > 0$ since $\boldsymbol{A}$ has rank n. The spectral condition number is $K = \sigma_1/\sigma_n$. We also use the Matlab function `diag(Sigma)` that extracts the diagonal of $\boldsymbol{\Sigma}$. This leads to the following program:

```
function [x,K]=lsq(A,b)
[U,Sigma,V]=svd(A,0);
s=diag(Sigma);
x=V*((U'*b)./s);
K=s(1)/s(length(s));
```

Good luck!