CHAPTER 7

Norms and Perturbation theory for linear systems

Exercise 7.7: Consistency of sum norm?

Observe that the sum norm is a matrix norm. This follows since it is equal to the $l_1$-norm of the vector $v = \text{vec}(A)$ obtained by stacking the columns of a matrix $A$ on top of each other.

Let $A = (a_{ij})_{ij}$ and $B = (b_{ij})_{ij}$ be matrices for which the product $AB$ is defined. Then

$$\|AB\|_S = \sum_{i,j} \left| \sum_k a_{ik} b_{kj} \right| \leq \sum_{i,j,k} |a_{ik}| \cdot |b_{kj}|$$

$$\leq \sum_{i,j,k,l} |a_{ik}| \cdot |b_{lj}| = \sum_{i,k} |a_{ik}| \sum_{l,j} |b_{lj}| = \|A\|_S \|B\|_S,$$

where the first inequality follows from the triangle inequality and multiplicative property of the absolute value $|\cdot|$. Since $A$ and $B$ where arbitrary, this proves that the sum norm is consistent.

Exercise 7.8: Consistency of max norm?

Observe that the max norm is a matrix norm. This follows since it is equal to the $l_\infty$-norm of the vector $v = \text{vec}(A)$ obtained by stacking the columns of a matrix $A$ on top of each other.

To show that the max norm is not consistent we use a counter example. Let $A = B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then

$$\|AB\|_M = \|\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\|_M = 2 > 1 = \left\| \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\|_M \left\| \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\|_M,$$

contradicting $\|AB\|_M \leq \|A\|_M \|B\|_M$.

Exercise 7.9: Consistency of modified max norm?

Exercise 7.8 shows that the max norm is not consistent. In this Exercise we show that the max norm can be modified so as to define a consistent matrix norm.

(a) Let $A \in \mathbb{C}^{m,n}$ and define $\|A\| := \sqrt{m \bar{n}} \|A\|_M$ as in the Exercise. To show that $\|\cdot\|$ defines a consistent matrix norm we have to show that it fulfills the three matrix norm properties and that it is submultiplicative. Let $A, B \in \mathbb{C}^{m,n}$ be any matrices and $\alpha$ any scalar.

(1) Positivity. Clearly $\|A\| = \sqrt{m \bar{n}} \|A\|_M \geq 0$. Moreover,

$$\|A\| = 0 \iff a_{i,j} = 0 \quad \forall i, j \iff A = 0.$$ 

(2) Homogeneity. $\|\alpha A\| = \sqrt{m \bar{n}} \|\alpha A\|_M = |\alpha| \sqrt{m \bar{n}} \|A\|_M = |\alpha| \|A\|$.
(3) Subadditivity. One has
\[ \|A + B\| = \sqrt{m n} \|A + B\|_M \leq \sqrt{m n} \left( \|A\|_M + \|B\|_M \right) = \|A\| + \|B\|. \]

(4) Submultiplicativity. One has
\[ \|AB\| = \sqrt{m n} \max_{1 \leq m \leq q} \sum_{k=1}^{q} |a_{i,k}b_{k,j}| \]
\[ \leq \sqrt{m n} \max_{1 \leq m \leq q} \sum_{k=1}^{q} |b_{k,j}| \max_{1 \leq k \leq q} \sum_{k=1}^{q} |a_{i,k}| \]
\[ \leq q \sqrt{m n} \left( \max_{1 \leq m \leq q} |a_{i,k}| \right) \left( \max_{1 \leq k \leq q} |b_{k,j}| \right) \]
\[ = \|A\| \|B\|. \]

(b) For any \( A \in \mathbb{C}^{m \times n} \), let
\[ \|A\|^{(1)} := m \|A\|_M \quad \text{and} \quad \|A\|^{(2)} := n \|A\|_M. \]

Comparing with the solution of part (a) we see, that the points of positivity, homogeneity and subadditivity are fulfilled here as well, making \( \|A\|^{(1)} \) and \( \|A\|^{(2)} \) valid matrix norms. Furthermore, for any \( A \in \mathbb{C}^{m \times q}, B \in \mathbb{C}^{q \times n} \),
\[ \|AB\|^{(1)} = m \max_{1 \leq m \leq q} \sum_{k=1}^{q} |a_{i,k}b_{k,j}| \leq m \left( \max_{1 \leq m \leq q} |a_{i,k}| \right) q \left( \max_{1 \leq k \leq q} |b_{k,j}| \right) \]
\[ = \|A\|^{(1)} \|B\|^{(1)}, \]
\[ \|AB\|^{(2)} = n \max_{1 \leq k \leq q} \sum_{k=1}^{q} |a_{i,k}b_{k,j}| \leq q \left( \max_{1 \leq m \leq q} |a_{i,k}| \right) n \left( \max_{1 \leq k \leq q} |b_{k,j}| \right) \]
\[ = \|A\|^{(2)} \|B\|^{(2)}, \]
which proves the submultiplicativity of both norms.

**Exercise 7.11**: The sum norm is subordinate to?

For any matrix \( A = (a_{ij})_{ij} \in \mathbb{C}^{m \times n} \) and column vector \( x = (x_j)_j \in \mathbb{C}^n \), one has
\[ \|Ax\|_1 = \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}x_j| \leq \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}| |x_j| \leq \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}| n \sum_{k=1}^{n} |x_k| = \|A\|_S \|x\|_1, \]
which shows that the matrix norm \( \|\cdot\|_S \) is subordinate to the vector norm \( \|\cdot\|_1 \).
values of
This proves the second claim. For the first claim, let Euclidean norm invariant,
the biggest singular value of
Let
which means that By (b), equality is attained for any standard basis vector
which is what needed to be shown.
(c) By (a), \( \|A\|_M \geq \|Ax\|_\infty / \|x\|_1 \) for all nonzero vectors \( x \), implying that
\[
\|A\|_M \geq \max_{x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_1}.
\]
By (b), equality is attained for any standard basis vector \( e_l \) for which there exists a \( k \) such that \( \|A\|_M = |a_{k,l}| \). We conclude that
\[
\|A\|_M = \max_{x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_1},
\]
which means that \( \cdot \|_M \) is the \((\infty, 1)\)-operator norm (see Definition 7.13).

**Exercise 7.19: Spectral norm**

Let \( A = UV^* \) be a singular value decomposition of \( A \), and write \( \sigma_1 := \|A\|_2 \) for the biggest singular value of \( A \). Since the orthogonal matrices \( U \) and \( V \) leave the Euclidean norm invariant,
\[
\max_{|x|_2 = 1 = |y|_2} |y^*Ax| = \max_{|x|_2 = 1 = |y|_2} |y^*U\Sigma V^*x| = \max_{|x|_2 = 1 = |y|_2} |y^*\Sigma x| = \max_{|x|_2 = 1 = |y|_2} \sigma_1 |y^*x| = \sigma_1 \|x\|_2 \|x\|_2 = \sigma_1.
\]
Moreover, this maximum is achieved for \( x = v_1 \) and \( y = u_1 \), and we conclude
\[
\|A\|_2 = \sigma_1 = \max_{|x|_2 = 1 = |y|_2} |y^*Ax|.
\]

**Exercise 7.20: Spectral norm of the inverse**

Since \( A \) is nonsingular we find
\[
\|A^{-1}\|_2 = \max_{x \neq 0} \frac{\|A^{-1}x\|_2}{\|x\|_2} = \max_{x \neq 0} \frac{\|x\|_2}{\|Ax\|_2}.
\]
This proves the second claim. For the first claim, let \( \sigma_1 \geq \cdots \geq \sigma_n \) be the singular values of \( A \). Again since \( A \) is nonsingular, \( \sigma_n \) must be nonzero, and Equation(7.17)
states that \( \frac{1}{\sigma_n} = \| A^{-1} \|_2 \). From this and what we just proved we have that \( \frac{1}{\sigma_n} \geq \frac{1}{\|Ax\|_2} \) for any \( x \) so that \( \|x\|_2 = 1 \), so that also \( \|Ax\|_2 \geq \sigma_n \) for such \( x \).

**Exercise 7.21: \( p \)-norm example**

We have

\[
A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad A^{-1} = \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.
\]

Using Theorem 7.15, one finds \( \|A\|_1 = \|A\|_\infty = 3 \) and \( \|A^{-1}\|_1 = \|A^{-1}\|_\infty = 1 \). The singular values \( \sigma_1 \geq \sigma_2 \) of \( A \) are the square roots of the zeros of

\[
0 = \det(A^T A - \lambda I) = (5 - \lambda)^2 - 16 = \lambda^2 - 10\lambda + 9 = (\lambda - 9)(\lambda - 1),
\]

Using Theorem 7.17, we find \( \|A\|_2 = \sigma_1 = 3 \) and \( \|A^{-1}\|_2 = \sigma_2^{-1} = 1 \). Alternatively, since \( A \) is symmetric positive definite, we know from (7.18) that \( \|A\|_2 = \lambda_1 \) and \( \|A^{-1}\|_2 = 1/\lambda_2 \), where \( \lambda_1 = 3 \) is the biggest eigenvalue of \( A \) and \( \lambda_2 = 1 \) is the smallest.

**Exercise 7.24: Unitary invariance of the spectral norm**

Suppose \( V \) is a rectangular matrix satisfying \( V^* V = I \). Then

\[
\|VA\|_2^2 = \max_{\|x\|_2 = 1} \|VAX\|_2^2 = \max_{\|x\|_2 = 1} x^* A^* V^* VAX
\]

\[
= \max_{\|x\|_2 = 1} x^* A^* AX = \max_{\|x\|_2 = 1} \|AX\|_2^2 = \|A\|_2^2.
\]

The result follows by taking square roots.

**Exercise 7.25: \( \|AU\|_2 \) rectangular \( A \)**

Let \( U = [u_1, u_2]^T \) be any \( 2 \times 1 \) matrix satisfying \( 1 = U^T U \). Then \( AU \) is a \( 2 \times 1 \)-matrix, and clearly the operator \( 2 \)-norm of a \( 2 \times 1 \)-matrix equals its euclidean norm (when viewed as a vector):

\[
\left\| \begin{bmatrix} a_1 \\ a_2 \\ x \end{bmatrix} \right\|_2 = \left\| \begin{bmatrix} a_1 x \\ a_2 x \end{bmatrix} \right\|_2 = \|x\| \left\| \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \right\|_2.
\]

In order for \( \|AU\|_2 < \|A\|_2 \) to hold, we need to find a vector \( v \) with \( \|v\|_2 = 1 \) so that \( \|AU\|_2 < \|Av\|_2 \). In other words, we need to pick a matrix \( A \) that scales more in the direction \( v \) than in the direction \( U \). For instance, if

\[
A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad v = \begin{bmatrix} 1 \\ 0 \end{bmatrix},
\]

then

\[
\|A\|_2 = \max_{\|x\|_2 = 1} \|AX\|_2 \geq \|AV\|_2 = 2 > 1 = \|AU\|_2.
\]
Exercise 7.26: $p$-norm of diagonal matrix

The eigenpairs of the matrix $A = \text{diag}(\lambda_1, \ldots, \lambda_n)$ are $(\lambda_1, e_1), \ldots, (\lambda_n, e_n)$. For $\rho(A) = \max\{|\lambda_1|, \ldots, |\lambda_n|\}$, one has

$$
\|A\|_p = \max_{(x_1, \ldots, x_n) \neq 0} \frac{|\lambda_1 x_1|^p + \cdots + |\lambda_n x_n|^p}{{(|x_1|^p + \cdots + |x_n|^p)}^{1/p}}
\leq \max_{(x_1, \ldots, x_n) \neq 0} \frac{(\rho(A))^p |x_1|^p + \cdots + (\rho(A))^p |x_n|^p}{(|x_1|^p + \cdots + |x_n|^p)^{1/p}} = \rho(A).
$$

On the other hand, for $x$ such that $\rho(A) = |\lambda_j|$, one finds

$$
\|A\|_p = \max_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p} = \frac{\|Ae_j\|_p}{\|e_j\|_p} = \rho(A).
$$

Together, the above two statements imply that $\|A\|_p = \rho(A)$ for any diagonal matrix $A$ and any $p$ satisfying $1 \leq p \leq \infty$.

Exercise 7.27: Spectral norm of a column vector

We write $A \in \mathbb{C}^{n \times 1}$ for the matrix corresponding to the column vector $a \in \mathbb{C}^n$. Write $\|A\|_p$ for the operator $p$-norm of $A$ and $\|a\|_p$ for the vector $p$-norm of $a$. In particular $\|A\|_2$ is the spectral norm of $A$ and $\|a\|_2$ is the Euclidean norm of $a$. Then

$$
\|A\|_p = \max_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p} = \max_{x \neq 0} \frac{|x||a||_p}{\|x\|} = \|a\|_p,
$$

proving (b). Note that (a) follows as the special case $p = 2$.

Exercise 7.28: Norm of absolute value matrix

(a) One finds

$$
|A| = \begin{bmatrix} 1+i & 1-2i \\ 1 & 1-i \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 2 \\ 1 & \sqrt{2} \end{bmatrix}.
$$

(b) Let $b_{i,j}$ denote the entries of $|A|$. Observe that $b_{i,j} = |a_{i,j}| = |b_{i,j}|$. Together with Theorem 7.15, these relations yield

$$
\|A\|_F = \left(\sum_{i=1}^m \sum_{j=1}^n |a_{i,j}|^2\right)^{1/2} = \left(\sum_{i=1}^m \sum_{j=1}^n |b_{i,j}|^2\right)^{1/2} = \|\text{vec}\ A\|_F,
$$

$$
\|A\|_1 = \max_{1 \leq j \leq n} \left(\sum_{i=1}^m |a_{i,j}|\right) = \max_{1 \leq j \leq n} \left(\sum_{i=1}^m |b_{i,j}|\right) = \|A\|_1,
$$

$$
\|A\|_\infty = \max_{1 \leq i \leq m} \left(\sum_{j=1}^n |a_{i,j}|\right) = \max_{1 \leq i \leq m} \left(\sum_{j=1}^n |b_{i,j}|\right) = \|A\|_\infty,
$$

which is what needed to be shown.

(c) To show this relation between the 2-norms of $A$ and $|A|$, we first examine the connection between the $l_2$-norms of $Ax$ and $|A||x|$, where $x = (x_1, \ldots, x_n)$ and $|x| = (|x_1|, \ldots, |x_n|)$. We find

$$
\|Ax\|_2 = \left(\sum_{i=1}^m \sum_{j=1}^n |a_{i,j}x_j|^2\right)^{1/2} \leq \left(\sum_{i=1}^m \left(\sum_{j=1}^n |a_{i,j}| |x_j|\right)^2\right)^{1/2} = \|A||x|\|_2.
$$
Now let \( \mathbf{x}^* \) with \( \| \mathbf{x}^* \|_2 = 1 \) be a vector for which \( \| \mathbf{A} \|_2 = \| \mathbf{A} \mathbf{x}^* \|_2 \). That is, let \( \mathbf{x}^* \) be a unit vector for which the maximum in the definition of 2-norm is attained. Observe that \( \| \mathbf{x}^* \|_2 \) is then a unit vector as well, \( \| \mathbf{x}^* \|_2 = 1 \). Then, by the above estimate of \( l_2 \)-norms and definition of the 2-norm,
\[
\| \mathbf{A} \|_2 = \| \mathbf{A} \mathbf{x}^* \|_2 \leq \| \mathbf{A} \| \| \mathbf{x}^* \|_2 \leq \| \mathbf{A} \|_2.
\]

(d) By Theorem 7.15, we can solve this exercise by finding a matrix \( \mathbf{A} \) for which \( \mathbf{A} \) and \( |\mathbf{A}| \) have different largest singular values. As \( \mathbf{A} \) is real and symmetric, there exist \( a, b, c \in \mathbb{R} \) such that
\[
\mathbf{A} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}, \quad |\mathbf{A}| = \begin{bmatrix} |a| & |b| \\ |b| & |c| \end{bmatrix},
\]
\[
\mathbf{A}^T \mathbf{A} = \begin{bmatrix} a^2 + b^2 & ab + bc \\ ab + bc & b^2 + c^2 \end{bmatrix}, \quad |\mathbf{A}^T| |\mathbf{A}| = \begin{bmatrix} a^2 + b^2 & |ab| + |bc| \\ |ab| + |bc| & b^2 + c^2 \end{bmatrix}.
\]
To simplify these equations we first try the case \( a + c = 0 \). Eliminating \( c \) we get
\[
\mathbf{A}^T \mathbf{A} = \begin{bmatrix} a^2 + b^2 & 0 \\ 0 & a^2 + b^2 \end{bmatrix}, \quad |\mathbf{A}^T| |\mathbf{A}| = \begin{bmatrix} a^2 + b^2 & 2|ab| \\ 2|ab| & 2(a^2 + b^2) \end{bmatrix}.
\]
To get different norms we have to choose \( a, b \) in such a way that the maximal eigenvalues of \( \mathbf{A}^T \mathbf{A} \) and \( |\mathbf{A}^T| |\mathbf{A}| \) are different. Clearly \( \mathbf{A}^T \mathbf{A} \) has a unique eigenvalue \( \lambda := a^2 + b^2 \) and putting the characteristic polynomial \( \pi(\mu) = (a^2 + b^2 - \mu)^2 - 4|ab|^2 \) of \( |\mathbf{A}^T| |\mathbf{A}| \) to zero yields eigenvalues \( \mu_\pm := a^2 + b^2 \pm 2|ab| \). Hence \( |\mathbf{A}^T| |\mathbf{A}| \) has maximal eigenvalue \( \mu_+ = a^2 + b^2 + 2|ab| = \lambda + 2|ab| \). The spectral norms of \( \mathbf{A} \) and \( |\mathbf{A}| \) therefore differ whenever both \( a \) and \( b \) are nonzero. For example, when \( a = b = -c = 1 \) we find
\[
\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \| \mathbf{A} \|_2 = \sqrt{2}, \quad \| |\mathbf{A}| \|_2 = 2.
\]

Exercise 7.35: Sharpness of perturbation bounds

Suppose \( \mathbf{A} \mathbf{x} = \mathbf{b} \) and \( \mathbf{A} \mathbf{y} = \mathbf{b} + \mathbf{e} \). Let \( K = K(\mathbf{A}) = \| \mathbf{A} \| \| \mathbf{A}^{-1} \| \) be the condition number of \( \mathbf{A} \). Let \( \mathbf{y}_\mathbf{A} \) and \( \mathbf{y}_{\mathbf{A}^{-1}} \) be unit vectors for which the maxima in the definition of the operator norms of \( \mathbf{A} \) and \( \mathbf{A}^{-1} \) are attained. That is, \( \| \mathbf{y}_\mathbf{A} \| = 1 = \| \mathbf{y}_{\mathbf{A}^{-1}} \| \), \( \| \mathbf{A} \| = \| \mathbf{A} \mathbf{y}_\mathbf{A} \| \), and \( \| \mathbf{A}^{-1} \| = \| \mathbf{A}^{-1} \mathbf{y}_{\mathbf{A}^{-1}} \| \). If \( \mathbf{b} = \mathbf{A} \mathbf{y}_\mathbf{A} \) and \( \mathbf{e} = \mathbf{y}_{\mathbf{A}^{-1}} \), then
\[
\frac{\| \mathbf{y} - \mathbf{x} \|}{\| \mathbf{x} \|} = \frac{\| \mathbf{A}^{-1} \mathbf{e} \|}{\| \mathbf{A}^{-1} \mathbf{b} \|} = \frac{\| \mathbf{A}^{-1} \mathbf{y}_{\mathbf{A}^{-1}} \|}{\| \mathbf{y}_\mathbf{A} \|} = \| \mathbf{A} \| \| \mathbf{A}^{-1} \| \| \mathbf{y}_{\mathbf{A}^{-1}} \| \| \mathbf{y}_\mathbf{A} \| = K \frac{\| \mathbf{e} \|}{\| \mathbf{b} \|},
\]
showing that the upper bound is sharp. If \( \mathbf{b} = \mathbf{y}_{\mathbf{A}^{-1}} \) and \( \mathbf{e} = \mathbf{A} \mathbf{y}_\mathbf{A} \), then
\[
\frac{\| \mathbf{y} - \mathbf{x} \|}{\| \mathbf{x} \|} = \frac{\| \mathbf{A}^{-1} \mathbf{e} \|}{\| \mathbf{A}^{-1} \mathbf{b} \|} = \frac{\| \mathbf{A} \mathbf{y}_\mathbf{A} \|}{\| \mathbf{A}^{-1} \mathbf{y}_{\mathbf{A}^{-1}} \|} = \frac{1}{\| \mathbf{A} \| \| \mathbf{A}^{-1} \| \| \mathbf{y}_{\mathbf{A}^{-1}} \|} = \frac{1}{K} \frac{\| \mathbf{e} \|}{\| \mathbf{b} \|},
\]
showing that the lower bound is sharp.

Exercise 7.36: Condition number of 2nd derivative matrix

Recall that \( \mathbf{T} = \text{tridiag}(-1, 2, -1) \) and, by Exercise 1.26, \( \mathbf{T}^{-1} \) is given by
\[
(T^{-1})_{ij} = (T^{-1})_{ji} = (1 - ih)j > 0, \quad 1 \leq j \leq i \leq m, \quad h = \frac{1}{m + 1}.
\]
From Theorems 7.15 and 7.17, we have the following explicit expressions for the 1-, 2- and $\infty$-norms

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^m |a_{ij}|, \quad \|A\|_2 = \sigma_1, \quad \|A^{-1}\|_2 = \frac{1}{\sigma_m}, \quad \|A\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^n |a_{ij}|$$

for any matrix $A \in \mathbb{C}^{m \times n}$, where $\sigma_1$ is the largest singular value of $A$, $\sigma_m$ is the smallest singular value of $A$, and we assumed $A$ to be nonsingular in the third equation.

a) For the matrix $T$ this gives $\|T\|_1 = \|T\|_\infty = m + 1$ for $m = 1, 2$ and $\|T\|_1 = \|T\|_\infty = 4$ for $m \geq 3$. For the inverse we get $\|T^{-1}\|_1 = \|T^{-1}\|_\infty = \frac{1}{2} = \frac{1}{2}h^{-2}$ for $m = 1$ and

$$\|T^{-1}\|_1 = \left\| \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right\|_1 = 1 = \left\| \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right\|_\infty = \|T^{-1}\|_\infty$$

for $m = 2$. For $m > 1$, one obtains

$$\sum_{i=1}^m \left| (T^{-1})_{ij} \right| = \sum_{i=1}^{j-1} (1 - jh)i + \sum_{i=j}^m (1 - ih)j$$

It is easy to commit an error when simplifying this sum. It can be computed symbolically using Symbolic Math Toolbox using the following code:

```matlab
syms i j m
simplify(symsum((1-j/(m+1)) * i, i, 1, j-1) + symsum((1-i/(m+1)) * j, i, j, m))
```

Here we have substituted $h = 1/(m + 1)$. This produces the result $\frac{1}{2}(m + 1 - j)$. To arrive at this ourselves, we can first rewrite the expression as

$$\sum_{i=1}^{j-1} (1 - jh)i + \sum_{i=1}^m (1 - ih)j - \sum_{i=1}^{j-1} (1 - ih)j.$$

The first sum here equals $(1 - jh)\frac{(j-1)j}{2}$. The second sum equals

$$\sum_{i=1}^m j - jh \sum_{i=1}^m i = mj - jh \frac{m(m + 1)}{2} = mj - mj/2 = mj/2.$$

The third sum equals

$$\sum_{i=1}^{j-1} j - jh \sum_{i=1}^{j-1} i = j(j-1) - hj^2\frac{j-1}{2} = j(j - 1)(2 - hj)/2.$$

Combining the three sums we get

$$\frac{j}{2} ((j - 1)(1 - jh) + m - (j-1)(2 - hj)) = \frac{j}{2}(m + 1 - j),$$

which we also arrived at above. This can also be written as $\frac{1}{2h}j - \frac{1}{2}j^2$ which is a quadratic function in $j$ that attains its maximum at $j = \frac{1}{2h} = \frac{m+1}{2}$. For odd $m > 1$, this function takes its maximum at integral $j$, yielding $\|T^{-1}\|_1 = \frac{1}{8}h^{-2}$. For even $m > 2$, on the other hand, the maximum over all integral $j$ is attained at $j = \frac{m}{2} = \frac{1-h}{2h}$ or $j = \frac{m+2}{2} = \frac{1+h}{2h}$, which both give $\|T^{-1}\|_1 = \frac{1}{8}(h^{-2} - 1).$
Since $T^{-1}$ is symmetric, the row sums equal the column sums, so that $\|T^{-1}\|_\infty = \|T^{-1}\|_1$. We conclude that the 1- and $\infty$-condition numbers of $T$ are:

$$\text{cond}_1(T) = \text{cond}_\infty(T) = \begin{cases} 2 & m = 1; \\ 6 & m = 2; \\ h^{-2} & m \text{ odd, } m > 1; \\ h^{-2} - 1 & m \text{ even, } m > 2. \end{cases}$$

b) Since the matrix $T$ is symmetric, $T^T T = T^2$ and the eigenvalues of $T^T T$ are the squares of the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $T$. As all eigenvalues of $T$ are positive, each singular value of $T$ is equal to an eigenvalue. Using that $\lambda_i = 2 - 2\cos(i\pi h)$, we find

$$\sigma_1 = |\lambda_m| = 2 - 2\cos(m\pi h) = 2 + 2\cos(\pi h),$$
$$\sigma_m = |\lambda_1| = 2 - 2\cos(\pi h).$$

It follows that

$$\text{cond}_2(T) = \frac{\sigma_1}{\sigma_m} = \frac{1 + \cos(\pi h)}{1 - \cos(\pi h)} = \cot^2 \left( \frac{\pi h}{2} \right).$$

c) From tan $x > x$ we obtain $\cot^2 x \geq \frac{1}{\tan^2 x} < \frac{1}{x^2}$. Using this and $\cot^2 x > x^{-2} - \frac{2}{3}$, we find

$$\frac{4}{\pi^2 h^2} - \frac{2}{3} < \text{cond}_2(T) < \frac{4}{\pi^2 h^2}.$$

d) For $p = 2$, substitute $h = 1/(m+1)$ in c) and use that $4/\pi^2 < 1/2$. For $p = 1, \infty$ we need to show due to a) that

$$\frac{4}{\pi^2 h^2} - 2/3 < 1/2 h^{-2} \leq \frac{1}{2} h^{-2},$$

when $m$ is odd, and that

$$\frac{4}{\pi^2 h^2} - 2/3 \leq \frac{1}{2} (h^{-2} - 1) \leq \frac{1}{2} h^{-2},$$

when $m$ is even. The right hand sides in these equations are obvious. The left equation for $m$ odd is also obvious since $4/\pi^2 < 1/2$. The left equation for $m$ even is also obvious since $-2/3 < -1/2$.

**Exercise 7.47:** When is a complex norm an inner product norm?

As in the Exercise, we let

$$\langle \mathbf{x}, \mathbf{y} \rangle = s(\mathbf{x}, \mathbf{y}) + i s(\mathbf{x}, i\mathbf{y}), \quad s(\mathbf{x}, \mathbf{y}) = \frac{\|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x} - \mathbf{y}\|^2}{4}.$$ 

We need to verify the three properties that define an inner product. Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be arbitrary vectors in $\mathbb{C}^m$ and $a \in \mathbb{C}$ be an arbitrary scalar.

1. **Positive-definiteness.** One has $s(\mathbf{x}, \mathbf{x}) = \|\mathbf{x}\|^2 \geq 0$ and

$$s(\mathbf{x}, i\mathbf{x}) = \frac{\|\mathbf{x} + i\mathbf{x}\|^2 - \|\mathbf{x} - i\mathbf{x}\|^2}{4} = \frac{\|(1 + i)\mathbf{x}\|^2 - \|(1 - i)\mathbf{x}\|^2}{4}$$

$$= \frac{(|1 + i| - |1 - i|)\|\mathbf{x}\|^2}{4} = 0,$$

so that $\langle \mathbf{x}, \mathbf{x} \rangle = \|\mathbf{x}\|^2 \geq 0$, with equality holding precisely when $\mathbf{x} = \mathbf{0}$.
(2) Conjugate symmetry. Since $\langle x, y \rangle$ is real, $\langle x, y \rangle = \langle y, x \rangle$, $s(ax, ay) = |a|^2 s(x, y)$, and $s(x, -y) = -s(x, y)$, 

$\langle y, x \rangle = s(y, x) - is(y, ix) = s(x, y) - is(ix, y) = s(x, y) - is(x, -iy) = \langle x, y \rangle$.

(3) Linearity in the first argument. Assuming the parallelogram identity,

$$2s(x, z) + 2s(y, z) = \frac{1}{2}||x + z||^2 - \frac{1}{2}||z - x||^2 + \frac{1}{2}||y + z||^2 - \frac{1}{2}||z - y||^2$$

$$= \frac{1}{2} \left( z + \frac{x + y}{2} + \frac{x - y}{2} \right)^2 - \frac{1}{2} \left( z - \frac{x + y}{2} - \frac{x - y}{2} \right)^2 +$$

$$+ \frac{1}{2} \left( z + \frac{x + y}{2} - \frac{x - y}{2} \right)^2 - \frac{1}{2} \left( z - \frac{x + y}{2} + \frac{x - y}{2} \right)^2$$

$$= \left| z + \frac{x + y}{2} \right|^2 + \left| z - \frac{x + y}{2} \right|^2 - \left| z - \frac{x + y}{2} \right|^2 - \left| z + \frac{x - y}{2} \right|^2$$

$$= 4s \left( \frac{x + y}{2}, z \right),$$

implying that $s(x + y, z) = s(x, z) + s(y, z)$. It follows that

$\langle x + y, z \rangle = s(x + y, z) + is(x + y, iz) = s(x, z) + s(y, z) + is(x, iz) + is(y, iz) = s(x, z) + is(x, iz) + s(y, z) + is(y, iz) = \langle x, z \rangle + \langle y, z \rangle$.

That $\langle ax, y \rangle = a \langle x, y \rangle$ follows, mutatis mutandis, from the proof of Theorem 7.45.

Exercise 7.48: p-norm for $p = 1$ and $p = \infty$

We need to verify the three properties that define a norm. Consider arbitrary vectors $x = [x_1, \ldots, x_n]^T$ and $y = [y_1, \ldots, y_n]$ in $\mathbb{R}^n$ and a scalar $a \in \mathbb{R}$. First we verify that $\| \cdot \|_1$ is a norm.

(1) Positivity. Clearly $\|x\|_1 = |x_1| + \cdots + |x_n| \geq 0$, with equality holding precisely when $|x_1| = \cdots = |x_n| = 0$, which happens if and only if $x$ is the zero vector.

(2) Homogeneity. One has

$$\|ax\|_1 = |ax_1| + \cdots + |ax_n| = |a|(|x_1| + \cdots + |x_n|) = |a|\|x\|_1.$$

(3) Subadditivity. Using the triangle inequality for the absolute value,

$$\|x+y\|_1 = |x_1+y_1| + \cdots + |x_n+y_n| \leq |x_1|+|y_1|+\cdots+|x_n|+|y_n| = \|x\|_1 + \|y\|_1.$$

Next we verify that $\| \cdot \|_\infty$ is a norm.

(1) Positivity. Clearly $\|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\} \geq 0$, with equality holding precisely when $|x_1| = \cdots = |x_n| = 0$, which happens if and only if $x$ is the zero vector.

(2) Homogeneity. One has

$$\|ax\|_\infty = \max\{|a||x_1|, \ldots, |a||x_n|\} = |a| \max\{|x_1|, \ldots, |x_n|\} = |a|\|x\|_\infty.$$
(3) Subadditivity. Using the triangle inequality for the absolute value,
\[ \|x + y\|_\infty = \max\{|x_1 + y_1|, \ldots, |x_n + y_n|\} \leq \max\{|x_1| + |y_1|, \ldots, |x_n| + |y_n|\} \]
\[ \leq \max\{|x_1|, \ldots, |x_n|\} + \max\{|y_1|, \ldots, |y_n|\} = \|x\|_\infty + \|y\|_\infty \]

Exercise 7.49: The \(p\)-norm unit sphere

In the plane, unit spheres for the 1-norm, 2-norm, and \(\infty\)-norm are

Exercise 7.50: Sharpness of \(p\)-norm inequality

Let 1 ≤ \(p\) ≤ \(\infty\). The vector \(x_i = [1, 0, \ldots, 0]^T \in \mathbb{R}^n\) satisfies
\[ \|x_i\|_p = (|1|^p + |0|^p + \cdots + |0|^p)^{1/p} = 1 = \max\{|1|, |0|, \ldots, |0|\} = \|x_i\|_\infty, \]
and the vector \(x_u = [1, 1, \ldots, 1]^T \in \mathbb{R}^n\) satisfies
\[ \|x_u\|_p = (|1|^p + \cdots + |1|^p)^{1/p} = n^{1/p} = n^{1/p} \max\{|1|, \ldots, |1|\} = n^{1/p}\|x_u\|_\infty. \]

Exercise 7.51: \(p\)-norm inequalities for arbitrary \(p\)

Let \(p\) and \(q\) be integers satisfying 1 ≤ \(q\) ≤ \(p\), and let \(x = [x_1, \ldots, x_n]^T \in \mathbb{C}^n\). Since \(p/q \geq 1\), the function \(f(z) = z^{p/q}\) is convex on \([0, \infty)\). For any \(z_1, \ldots, z_n \in [0, \infty)\) and \(\lambda_1, \ldots, \lambda_n \geq 0\) satisfying \(\lambda_1 + \cdots + \lambda_n = 1\), Jensen’s inequality gives
\[ \left( \sum_{i=1}^{n} \lambda_i z_i \right)^{p/q} = f \left( \sum_{i=1}^{n} \lambda_i z_i \right) \leq \sum_{i=1}^{n} \lambda_i f(z_i) = \sum_{i=1}^{n} \lambda_i z_i^{p/q}. \]
In particular for \(z_i = |x_i|^q\) and \(\lambda_1 = \cdots = \lambda_n = 1/n\),
\[ n^{-p/q} \left( \sum_{i=1}^{n} |x_i|^q \right)^{p/q} = \left( \sum_{i=1}^{n} \frac{1}{n} |x_i|^q \right)^{p/q} \leq \sum_{i=1}^{n} \frac{1}{n} (|x_i|^q)^{p/q} = n^{-1} \sum_{i=1}^{n} |x_i|^p. \]
Since the function \(x \mapsto x^{1/p}\) is monotone, we obtain
\[ n^{-1/q}\|x\|_q = n^{-1/q} \left( \sum_{i=1}^{n} |x_i|^q \right)^{1/p} \leq n^{-1/p} \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/q} = n^{-1/p}\|x\|_p, \]
from which the right inequality in the exercise follows.

The left inequality clearly holds for \(x = 0\), so assume \(x \neq 0\). Without loss of generality we can then assume \(\|x\|_\infty = 1\), since \(\|ax\|_p \leq \|ax\|_q\) if and only if \(\|x\|_p \leq \|x\|_q\) for any nonzero scalar \(a\). Then, for any \(i = 1, \ldots, n\), one has \(|x_i| \leq 1\), implying
that $|x_i|^p \leq |x_i|^q$. Moreover, since $|x_i| = 1$ for some $i$, one has $|x_1|^q + \cdots + |x_n|^q \geq 1$, so that

$$
\|x\|_p = \left( \sum_{i=1}^n |x_i|^p \right)^{1/p} \leq \left( \sum_{i=1}^n |x_i|^q \right)^{1/p} \leq \left( \sum_{i=1}^n |x_i|^q \right)^{1/q} = \|x\|_q.
$$

Finally we consider the case $p = \infty$. The statement is obvious for $q = p$, so assume that $q$ is an integer. Then

$$
\|x\|_q = \left( \sum_{i=1}^n |x_i|^q \right)^{1/q} \leq \left( \sum_{i=1}^n \|x\|_{\infty}^q \right)^{1/q} = n^{1/q} \|x\|_{\infty},
$$

proving the right inequality. Using that the map $x \mapsto x^{1/q}$ is monotone, the left inequality follows from

$$
\|x\|_\infty = (\max_i |x_i|)^q \leq \sum_{i=1}^n |x_i|^q = \|x\|_q^q.
$$

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