UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in INF-MAT3350/4350 — Numerical Linear Algebra

Day of examination: 06 December 2006

Examination hours: 09.00 – 12.00

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Each subproblem has a weight that is written in parenthesis.

Problem 1 Yes or No

Answer simply yes or no to the following questions:

1a (2)

Every matrix $\boldsymbol{A} \in \mathbb{C}^{m,n}$ has a singular value decomposition?

1b (2)

The algebraic multiplicity of an eigenvalue is always less than or equal to the geometric multiplicity?

1c (2)

The QR factorization of a matrix $\boldsymbol{A} \in \mathbb{R}^{n,n}$ can be determined by Householder transformations in $O(n^2)$ flops?

1d (2)

Let$\rho(\boldsymbol{A})$ be the spectral radius of $\boldsymbol{A} \in \mathbb{C}^{n,n}$. Then $\lim_{k\to\infty} \boldsymbol{A}^k = \boldsymbol{0}$ if and only if $\rho(\boldsymbol{A}) < 1$.

Problem 2 QR-factorization of band matrices

Let $\boldsymbol{A} \in \mathbb{R}^{n,n}$ be a nonsingular symmetric band matrix with bandwidth $d \le n-1$, so that $a_{ij} = 0$ for all i, j with $|i-j| > d$. Vi define $\boldsymbol{B} := \boldsymbol{A}^T\boldsymbol{A}$

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and let $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{R}$ be the QR-factorization of $\boldsymbol{A}$ where $\boldsymbol{R}$ has positive diagonal entries.

2a (2)

Show that $\boldsymbol{B}$ is symmetric.

2b (8)

Show that $\boldsymbol{B}$ has bandwidth $\leq 2d$.

2c (10)

Write a MATLAB `function B=ata(A,d)` which computes $\boldsymbol{B}$. You shall exploit the symmetry and the function should only use $O(cn^2)$ flops, where c only depends on d.

2d (5)

Estimate the number of flops in your algorithm.

2e (2)

Show that $\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{R}^T\boldsymbol{R}$.

2f (8)

Explain why $\boldsymbol{R}$ has upper bandwidth $2d$.

2g (10)

We consider 3 methods for finding the QR-factorization of the band matrix $\boldsymbol{A}$, where we assume that n is much bigger than d. The methods are based on

1. Gram-Schmidt orthogonalization,
2. Householder transformations,
3. Givens rotations.

Which method would you recommend for a computer program using floating point arithmetic? Give reasons for your answer.

Problem 3 Diagonalization and eigenvectors

Given the matrices $\boldsymbol{A} \in \mathbb{R}^{n,n}$ and $\boldsymbol{B} = \boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S}$, where $\boldsymbol{S} \in \mathbb{R}^{n,n}$ is nonsingular.

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3a (4)

Show that $(\lambda, \boldsymbol{x})$ is an eigenpair for $\boldsymbol{B}$ if and only if $(\lambda, \boldsymbol{Sx})$ is an eigenpair for $\boldsymbol{A}$.

3b (8)

The columns in $\boldsymbol{S}$ are eigenvectors for $\boldsymbol{A}$ if and only if $\boldsymbol{B}$ is a diagonal matrix.

Good luck!