UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in INF-MAT4350 — Numerical linear algebra

Day of examination: 3 December 2009

Examination hours: 0900 – 1200

This problem set consists of 2 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All 7 part questions will be weighted equally.

Problem 1 Matrix products

Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{E} \in \mathbb{R}^{n,n}$ be matrices where $\boldsymbol{A}^T = \boldsymbol{A}$. In this problem an (arithmetic) operation is an addition or a multiplication. We ask about exact numbers of operations.

1a

How many operations are required to compute the matrix product $\boldsymbol{BC}$? How many operations are required if $\boldsymbol{B}$ is lower triangular?

1b

Show that there exists a lower triangular matrix $\boldsymbol{L} \in \mathbb{R}^{n,n}$ such that $\boldsymbol{A} = \boldsymbol{L} + \boldsymbol{L}^T$.

1c

We have $\boldsymbol{E}^T\boldsymbol{A}\boldsymbol{E} = \boldsymbol{S} + \boldsymbol{S}^T$ where $\boldsymbol{S} = \boldsymbol{E}^T\boldsymbol{L}\boldsymbol{E}$. How many operations are required to compute $\boldsymbol{E}^T\boldsymbol{A}\boldsymbol{E}$ in this way?

Problem 2 Gershgorin Disks

The eigenvalues of $\boldsymbol{A} \in \mathbb{R}^{n,n}$ lie inside $R \cap C$, where $R := R_1 \cup \cdots \cup R_n$ is the union of the row disks R_i of $\boldsymbol{A}$, and $C = C_1 \cup \cdots \cup C_n$ is the union of the column disks C_j. You do not need to prove this. Write a Matlab function

(Continued on page 2.)

`[s,r,c]=gershgorin(A)` that computes the centres $\boldsymbol{s} = [s_1, \ldots, s_n] \in \mathbb{R}^n$ of the row and column disks, and their radii $\boldsymbol{r} = [r_1, \ldots, r_n] \in \mathbb{R}^n$ and $\boldsymbol{c} = [c_1, \ldots, c_n] \in \mathbb{R}^n$, respectively.

Problem 3 Eigenpairs

Let $\boldsymbol{A} \in \mathbb{R}^{n,n}$ be tridiagonal (i.e. $a_{ij} = 0$ when $|i-j| > 1$) and suppose also that $a_{i+1,i}a_{i,i+1} > 0$ for $i = 1, \ldots, n-1$.

3a

Show that for an arbitrary nonsingular diagonal matrix $\boldsymbol{D} = \mathrm{diag}(d_1, d_2, \ldots, d_n) \in \mathbb{R}^{n,n}$, the matrix

$$\boldsymbol{B} = \boldsymbol{D}^{-1}\boldsymbol{A}\boldsymbol{D} \tag{1}$$

is tridiagonal by finding a formula for b_{ij}, $i, j = 1, \ldots, n$.

3b

Show that there exists a choice of $\boldsymbol{D}$ such that $\boldsymbol{B}$ is symmetric and determine b_{ii} for $i = 1, \ldots, n$ and $b_{i,i+1}$ for $i = 1, \ldots, n-1$ with the choice $d_1 = 1$.

3c

Show that $\boldsymbol{B}$ and $\boldsymbol{A}$ have the same characteristic polynomials and explain why $\boldsymbol{A}$ has real eigenvalues.

Good luck!