# UNIVERSITY OF OSLO

## Faculty of mathematics and natural sciences

Examination in INF-MAT4350 — Numerical linear algebra

Day of examination: 8 December 2010

Examination hours: 0900 – 1300

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All 8 part questions will be weighted equally.

## Problem 1 Householder transformations

### 1a

Suppose $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ with $\|\boldsymbol{x}\|_2 = \|\boldsymbol{y}\|_2$ and $\boldsymbol{v} := \boldsymbol{x} - \boldsymbol{y} \neq 0$. Show that

$$\boldsymbol{H}\boldsymbol{x} = \boldsymbol{y}, \quad \text{where} \quad \boldsymbol{H} := \boldsymbol{I} - 2\frac{\boldsymbol{v}\boldsymbol{v}^T}{\boldsymbol{v}^T\boldsymbol{v}}.$$

### 1b

Let $\boldsymbol{B} \in \mathbb{R}^{4,4}$ be given by

$$\boldsymbol{B} := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \epsilon & 0 & 0 & 0 \end{bmatrix}, \tag{1}$$

where $0 < \epsilon < 1$. Compute a Householder transformation $\boldsymbol{H}$ and a matrix $\boldsymbol{B}_1$ such that the first column of $\boldsymbol{B}_1 := \boldsymbol{H}\boldsymbol{B}\boldsymbol{H}$ has a zero in the last two positions.

## Problem 2 Eigenvalue perturbations

Let $\boldsymbol{A} = [a_{kj}]$, $\boldsymbol{E} = [e_{kj}]$, and $\boldsymbol{B} = [b_{kj}]$ be matrices in $\mathbb{R}^{n,n}$ with

$$a_{kj} = \begin{cases} 1, & j = k+1, \\ 0, & \text{otherwise}, \end{cases} \qquad e_{kj} = \begin{cases} \epsilon, & k = n, j = 1, \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

(Continued on page 2.)

and $\boldsymbol{B} = \boldsymbol{A} + \boldsymbol{E}$, where $0 < \epsilon < 1$. Thus for $n = 4$,

$$\boldsymbol{A} := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{E} := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \epsilon & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{B} := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \epsilon & 0 & 0 & 0 \end{bmatrix}.$$

### 2a

Find the eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{B}$.

### 2b

Show that $\|\boldsymbol{A}\|_2 = \|\boldsymbol{B}\|_2 = 1$ for arbitrary $n \in \mathbb{N}$.

### 2c

Consider the following theorem (do not prove it!):

**Theorem**[Elsner's Theorem] Suppose $\boldsymbol{A}, \boldsymbol{E} \in \mathbb{C}^{n,n}$. To every eigenvalue $\mu$ of $\boldsymbol{A} + \boldsymbol{E}$ there is an eigenvalue $\lambda$ of $\boldsymbol{A}$ such that

$$|\mu - \lambda| \leq \left(\|\boldsymbol{A}\|_2 + \|\boldsymbol{A} + \boldsymbol{E}\|_2\right)^{1-1/n} \|\boldsymbol{E}\|_2^{1/n}. \tag{3}$$

Let $\boldsymbol{A}, \boldsymbol{E}, \boldsymbol{B}$ be given by (2). What upper bound does (3) give for the eigenvalue $\mu = \epsilon^{1/n}$ of $\boldsymbol{B}$? How sharp is this upper bound?

## Problem 3 The one norm

For $\boldsymbol{A} \in \mathbb{C}^{m,n}$ with $m, n \geq 1$ the one norm is defined by

$$\|\boldsymbol{A}\|_1 := \max_{\|\boldsymbol{x}\|_1 = 1} \|\boldsymbol{A}\boldsymbol{x}\|_1.$$

Show that

$$\|\boldsymbol{A}\|_1 = \max_{1 \leq j \leq n} \sum_{k=1}^{m} |a_{k,j}|. \tag{4}$$

## Problem 4 Hadamard's Inequaltiy

Let $\boldsymbol{A} \in \mathbb{C}^{n,n}$ be a general square matrix and let $\boldsymbol{A} = [\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n]$ be its column partition. Prove Hadamard's inequality,

$$\det(\boldsymbol{A}) \leq \prod_{j=1}^{n} \|\boldsymbol{a}_j\|_2.$$

Hint: Use a QR factorization of $\boldsymbol{A}$.

*(Continued on page 3.)*

## Problem 5 Matlab

Write a Matlab routine to solve the equation $\boldsymbol{Ax} = \boldsymbol{b}$ for $\boldsymbol{x} \in \mathbb{R}^n$, when $\boldsymbol{A} \in \mathbb{R}^{n,n}$ is a non-singular upper triangular matrix and $\boldsymbol{b} \in \mathbb{R}^n$.

Good luck!