UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in INF-MAT 4350 — Numerical linear algebra

Day of examination: 7 December 2011

Examination hours: 0900 – 1300

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All 8 part questions will be weighted equally.

Problem 1 Method of steepest descent

The method of steepest descent can be used to solve a linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ for $\boldsymbol{x} \in \mathbb{R}^n$, where $\boldsymbol{A} \in \mathbb{R}^{n,n}$ is symmetric and positive definite, and $\boldsymbol{b} \in \mathbb{R}^n$. With $\boldsymbol{x}_0 \in \mathbb{R}^n$ an initial guess, the iteration is

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \alpha_k \boldsymbol{r}_k,$$

where $\boldsymbol{r}_k$ is the residual, $\boldsymbol{r}_k = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}_k$, and

$$\alpha_k = \frac{\boldsymbol{r}_k^T \boldsymbol{r}_k}{\boldsymbol{r}_k^T \boldsymbol{A} \boldsymbol{r}_k}.$$

1a

Compute $\boldsymbol{x}_1$ if $\boldsymbol{A} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $\boldsymbol{b} = [1\ 1]^T$ and $\boldsymbol{x}_0 = \boldsymbol{0}$.

1b

If the k-th error, $\boldsymbol{e}_k = \boldsymbol{x}_k - \boldsymbol{x}$, is an eigenvector of $\boldsymbol{A}$, what can you say about $\boldsymbol{x}_{k+1}$?

Problem 2 Polar Decomposition

Given $n \in \mathbb{N}$ and a singular value decomposition $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$ of a square matrix $\boldsymbol{A} \in \mathbb{R}^{n,n}$, consider the matrices

$$\boldsymbol{Q} := \boldsymbol{U}\boldsymbol{V}^T, \quad \boldsymbol{P} := \boldsymbol{V}\boldsymbol{\Sigma}\boldsymbol{V}^T \tag{1}$$

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of order n.

2a

Show that
$$\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{P} \tag{2}$$
and show that $\boldsymbol{Q}$ is orthonormal.

2b

Show that $\boldsymbol{P}$ is symmetric positive semidefinite and positive definite if $\boldsymbol{A}$ is nonsingular.

The factorization in (2) is called a **polar factorization** of $\boldsymbol{A}$.

2c

Use the singular value decomposition of $\boldsymbol{A}$ to give a suitable definition of $\boldsymbol{B} := \sqrt{\boldsymbol{A}^T\boldsymbol{A}}$ so that $\boldsymbol{P} = \boldsymbol{B}$.

For the rest of this problem assume that $\boldsymbol{A}$ is nonsingular. Consider the iterative method
$$\boldsymbol{X}_{k+1} = \frac{1}{2}\big(\boldsymbol{X}_k + \boldsymbol{X}_k^{-T}\big),\ k = 0, 1, 2, \ldots \text{ with } \boldsymbol{X}_0 = \boldsymbol{A}, \tag{3}$$
for finding $\boldsymbol{Q}$.

2d

Show that the iteration (3) is well defined by showing that $\boldsymbol{X}_k = \boldsymbol{U}\boldsymbol{\Sigma}_k\boldsymbol{V}^T$, for a diagonal matrix Σ_k with positive diagonal elements, $k = 0, 1, 2, \ldots$.

2e

Show that
$$\boldsymbol{X}_{k+1} - \boldsymbol{Q} = \frac{1}{2}\boldsymbol{X}_k^{-T}\big(\boldsymbol{X}_k^T - \boldsymbol{Q}^T\big)\big(\boldsymbol{X}_k - \boldsymbol{Q}\big) \tag{4}$$
and use (4) and the Frobenius norm to show (quadratic convergence to $\boldsymbol{Q}$)
$$\|\boldsymbol{X}_{k+1} - \boldsymbol{Q}\|_F \le \frac{1}{2}\|\boldsymbol{X}_k^{-1}\|_F\|\boldsymbol{X}_k - \boldsymbol{Q}\|_F^2. \tag{5}$$

(Continued on page 3.)

2f

Write a Matlab program `function [Q,P,k] = polardecomp(A,tol,K)` to carry out the iteration in (3). The output is approximations $\boldsymbol{Q}$ and $\boldsymbol{P} = \boldsymbol{Q}^T\boldsymbol{A}$ to the polar decomposition $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{P}$ of $\boldsymbol{A}$ and the number of iterations k such that $\|\boldsymbol{X}_{k+1} - \boldsymbol{X}_k\|_F < tol * \|\boldsymbol{X}_{k+1}\|_F$. Set $k = K + 1$ if convergence is not achieved in K iterations. The Frobenius norm in Matlab is written `norm(A,'fro')`.

Good luck!