UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in	INF-MAT4350 — Numerical linear algebra
Day of examination:	8 December 2010
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This problem set consists of 4 pages.

Appendices:	None
Permitted aids:	None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All 8 part questions will be weighted equally.

Problem 1 Householder transformations

1a

Suppose $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ with $\|\boldsymbol{x}\|_2 = \|\boldsymbol{y}\|_2$ and $\boldsymbol{v} := \boldsymbol{x} - \boldsymbol{y} \neq 0$. Show that

$$\boldsymbol{H}\boldsymbol{x} = \boldsymbol{y}, \quad \text{where} \quad \boldsymbol{H} := \boldsymbol{I} - 2\frac{\boldsymbol{v}\boldsymbol{v}^T}{\boldsymbol{v}^T\boldsymbol{v}}.$$

Answer: Since

$$\boldsymbol{v}^T\boldsymbol{v} = (\boldsymbol{x}-\boldsymbol{y})^T(\boldsymbol{x}-\boldsymbol{y}) = \boldsymbol{x}^T\boldsymbol{x} - 2\boldsymbol{x}^T\boldsymbol{y} + \boldsymbol{y}^T\boldsymbol{y} = 2\boldsymbol{x}^T\boldsymbol{x} - 2\boldsymbol{x}^T\boldsymbol{y} = 2\boldsymbol{x}^T\boldsymbol{v} = 2\boldsymbol{v}^T\boldsymbol{x},$$

$$\boldsymbol{H}\boldsymbol{x} = (\boldsymbol{I} - 2\frac{\boldsymbol{v}\boldsymbol{v}^T}{\boldsymbol{v}^T\boldsymbol{v}})\boldsymbol{x} = \boldsymbol{x} - \frac{2\boldsymbol{v}^T\boldsymbol{x}}{2\boldsymbol{v}^T\boldsymbol{x}}\boldsymbol{v} = \boldsymbol{x} - (\boldsymbol{x} - \boldsymbol{y}) = \boldsymbol{y}.$$

1b

Let $\boldsymbol{B} \in \mathbb{R}^{4,4}$ be given by

$$\boldsymbol{B} := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \epsilon & 0 & 0 & 0 \end{bmatrix}, \tag{1}$$

where $0 < \epsilon < 1$. Compute a Householder transformation $\boldsymbol{H}$ and a matrix $\boldsymbol{B}_1$ such that the first column of $\boldsymbol{B}_1 := \boldsymbol{H}\boldsymbol{B}\boldsymbol{H}$ has a zero in the last two positions.

(Continued on page 2.)

Answer: Let $\boldsymbol{V} \in \mathbb{R}^{3,3}$ be a Householder transformation mapping $\boldsymbol{x} := [0,0,\epsilon]^T$ into $\boldsymbol{y} = [\epsilon,0,0]^T$. With $\boldsymbol{v} = \boldsymbol{x} - \boldsymbol{y} = \epsilon[-1,0,1]^T$ we find

$$\boldsymbol{V} = \boldsymbol{I} - 2\frac{\boldsymbol{v}\boldsymbol{v}^T}{\boldsymbol{v}^T\boldsymbol{v}} = \boldsymbol{I} - 2\epsilon^2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} /(2\epsilon^2) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Let

$$\boldsymbol{H} = \begin{bmatrix} 1 & \boldsymbol{0}^T \\ \boldsymbol{0} & \boldsymbol{V} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Then

$$\boldsymbol{B}_1 = \boldsymbol{H}\boldsymbol{B}\boldsymbol{H} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ \epsilon & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

This matrix is upper Hessenberg and has the same eigenvalues as $\boldsymbol{B}$ since it is similar to $\boldsymbol{B}$.

Problem 2 Eigenvalue perturbations

Let $\boldsymbol{A} = [a_{kj}]$, $\boldsymbol{E} = [e_{kj}]$, and $\boldsymbol{B} = [b_{kj}]$ be matrices in $\mathbb{R}^{n,n}$ with

$$a_{kj} = \begin{cases} 1, & j = k+1, \\ 0, & \text{otherwise}, \end{cases} \quad e_{kj} = \begin{cases} \epsilon, & k = n, j = 1, \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

and $\boldsymbol{B} = \boldsymbol{A} + \boldsymbol{E}$, where $0 < \epsilon < 1$. Thus for $n = 4$,

$$\boldsymbol{A} := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{E} := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \epsilon & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{B} := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \epsilon & 0 & 0 & 0 \end{bmatrix}.$$

2a

Find the eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{B}$.
Answer: Since $\boldsymbol{A}$ is triangular its eigenvalues λ_j are its diagonal elements. Thus the eigenvalues of $\boldsymbol{A}$ are $\lambda_j = 0$, $j = 1,2,\ldots,n$.

For $\boldsymbol{B}$ the equation $\boldsymbol{B}\boldsymbol{x} = \lambda\boldsymbol{x}$ gives $\lambda x_k = x_{k+1}$, $k = 1,\ldots,n-1$ and $\lambda x_n = \epsilon x_1$. Thus

$$\lambda^n x_n = \epsilon\lambda^{n-1}x_1 = \epsilon\lambda^{n-2}x_2 = \epsilon\lambda^{n-3}x_3 = \cdots = \epsilon x_n.$$

We must have $x_n \neq 0$ since otherwise $\boldsymbol{x} = \boldsymbol{0}$. Canceling x_n we find $\lambda^n = \epsilon$ and the eigenvalues of $\boldsymbol{B}$ are

$$\mu_j = \mu e^{2\pi i j/n}, \quad j = 1,\ldots,n,$$

where $\mu = \epsilon^{1/n}$ and $i = \sqrt{-1}$. Alternatively, expanding the determinant of $\boldsymbol{B}$ by its first column yields its characteristic polynomial, $\pi_{\boldsymbol{B}}(\lambda) = (-1)^n(\lambda^n - \epsilon)$ leading to the same eigenvalues.

(Continued on page 3.)

2b

Show that $\|\boldsymbol{A}\|_2 = \|\boldsymbol{B}\|_2 = 1$ for arbitrary $n \in \mathbb{N}$.
Answer: The spectral norm of $\boldsymbol{A}$ is the largest singular value which is the square root of the largest eigenvalue of $\boldsymbol{A}^T\boldsymbol{A}$. Now $\boldsymbol{A}^T\boldsymbol{A} = \mathrm{diag}(0, 1, \ldots, 1)$ is diagonal with eigenvalues 0 and 1. Thus $\|\boldsymbol{A}\|_2 = 1$. For $\boldsymbol{B}$ we find $\boldsymbol{B}^T\boldsymbol{B} = \mathrm{diag}(\epsilon^2, 1, \ldots, 1)$ and $\|\boldsymbol{B}\|_2 = 1$.

2c

Consider the following theorem (do not prove it!):

Theorem[Elsner's Theorem] Suppose $\boldsymbol{A}, \boldsymbol{E} \in \mathbb{C}^{n,n}$. To every eigenvalue μ of $\boldsymbol{A} + \boldsymbol{E}$ there is an eigenvalue λ of $\boldsymbol{A}$ such that

$$|\mu - \lambda| \leq \big(\|\boldsymbol{A}\|_2 + \|\boldsymbol{A} + \boldsymbol{E}\|_2\big)^{1-1/n} \|\boldsymbol{E}\|_2^{1/n}. \tag{3}$$

Let $\boldsymbol{A}, \boldsymbol{E}, \boldsymbol{B}$ be given by (2). What upper bound does (3) give for the eigenvalue $\mu = \epsilon^{1/n}$ of $\boldsymbol{B}$? How sharp is this upper bound?
Answer: We have $\|\boldsymbol{A}\|_2 = \|\boldsymbol{B}\|_2 = 1$. Now $\boldsymbol{E}^T\boldsymbol{E} = diag(\epsilon^2, 0, \ldots, 0)$ so $\|\boldsymbol{E}\|_2 = \epsilon$. Elsner's theorem says that there is an eigenvalue λ of $\boldsymbol{A}$ such that

$$|\mu - \lambda| \leq \big(\|\boldsymbol{A}\|_2 + \|\boldsymbol{B}\|_2\big)^{1-1/n} \|\boldsymbol{E}\|_2^{1/n} = 2^{1-1/n}\epsilon^{1/n} \leq 2\epsilon^{1/n}.$$

Since $\lambda = 0$ the exact difference is

$$|\mu - \lambda| = \mu = \epsilon^{1/n}.$$

The upper bound differ from the exact value by a constant less than 2, so it is quite sharp. It captures the $\epsilon^{1/n}$ behavior.

Problem 3 The one norm

For $\boldsymbol{A} \in \mathbb{C}^{m,n}$ with $m, n \geq 1$ the one norm is defined by

$$\|\boldsymbol{A}\|_1 := \max_{\|\boldsymbol{x}\|_1 = 1} \|\boldsymbol{A}\boldsymbol{x}\|_1.$$

Show that

$$\|\boldsymbol{A}\|_1 = \max_{1 \leq j \leq n} \sum_{k=1}^{m} |a_{k,j}|. \tag{4}$$

Answer: Let $\boldsymbol{x} \in \mathbb{C}^n$ with $\|\boldsymbol{x}\|_1 = 1$. Then

$$\|\boldsymbol{A}\boldsymbol{x}\|_1 = \sum_{k=1}^{m} \big|\sum_{j=1}^{n} a_{kj}x_j\big| \leq \sum_{k=1}^{m}\sum_{j=1}^{n} |a_{kj}||x_j| = \sum_{j=1}^{n} \big(\sum_{k=1}^{m} |a_{kj}|\big)|x_j| \leq \max_{1 \leq j \leq n} \sum_{k=1}^{m} |a_{k,j}|.$$

Let $j = c$ be the column where the max in (4) is attained. Then $\|\boldsymbol{A}\boldsymbol{e}_c\|_1 = \sum_{k=1}^{m} |a_{k,c}|$ so we have equality in (4) for $\boldsymbol{x} = \boldsymbol{e}_c$.

(Continued on page 4.)

Problem 4 Hadamard's Inequaltiy

Let $\boldsymbol{A} \in \mathbb{C}^{n,n}$ be a general square matrix and let $\boldsymbol{A} = [\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n]$ be its column partition. Prove Hadamard's inequality,

$$\det(\boldsymbol{A}) \leq \prod_{j=1}^{n} \|\boldsymbol{a}_j\|_2.$$

Hint: Use a QR factorization of $\boldsymbol{A}$.
Answer: Let $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{R}$ be a QR factorization of $\boldsymbol{A}$. Since

$$1 = \det(\boldsymbol{I}) = \det(\boldsymbol{Q}^*\boldsymbol{Q}) = \det(\boldsymbol{Q}^*)\det(\boldsymbol{Q}) = |\det(\boldsymbol{Q})|^2,$$

we have $|\det(\boldsymbol{Q})| = 1$. Let $\boldsymbol{R} = [\boldsymbol{r}_1, \ldots, \boldsymbol{r}_n]$. Then $\|\boldsymbol{a}_j\|_2 = \|\boldsymbol{r}_j\|_2$ because $\boldsymbol{a}_j = \boldsymbol{Q}\boldsymbol{r}_j$ and $\boldsymbol{Q}^*\boldsymbol{Q} = \boldsymbol{I}$. Therefore,

$$|\det(\boldsymbol{A})| = |\det(\boldsymbol{Q}\boldsymbol{R})| = |\det(\boldsymbol{R})| = \prod_{j=1}^{n} |r_{jj}| \leq \prod_{j=1}^{n} \|\boldsymbol{r}_j\|_2 = \prod_{j=1}^{n} \|\boldsymbol{a}_j\|_2.$$

Problem 5 Matlab

Write a Matlab routine to solve the equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ for $\boldsymbol{x} \in \mathbb{R}^n$, when $\boldsymbol{A} \in \mathbb{R}^{n,n}$ is a non-singular upper triangular matrix and $\boldsymbol{b} \in \mathbb{R}^n$.

Answer: The algorithm to find $\boldsymbol{x}$ is to let, for $i = n, n-1, \ldots, 1$, $x_i = (b_i - \sum_{j=i+1}^{n} a_{ij}x_j)/a_{ii}$, which is known as back substitution. A Matlab routine to do this is

```
function x = backsubst(A,b)
  n = length(b);
  x = b;
  for i=n:-1:1
     x(i) = (x(i)-A(i,i+1:n)*x(i+1:n))/A(i,i);
  end
```

Good luck!