UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in	INF-MAT 4350 — Numerical linear algebra
Day of examination:	7 December 2011
Examination hours:	0900 – 1300
This problem set consists of 4 pages.	
Appendices:	None
Permitted aids:	None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All 8 part questions will be weighted equally.

Problem 1 Method of steepest descent

The method of steepest descent can be used to solve a linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ for $\boldsymbol{x} \in \mathbb{R}^n$, where $\boldsymbol{A} \in \mathbb{R}^{n,n}$ is symmetric and positive definite, and $\boldsymbol{b} \in \mathbb{R}^n$. With $\boldsymbol{x}_0 \in \mathbb{R}^n$ an initial guess, the iteration is

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \alpha_k \boldsymbol{r}_k,$$

where $\boldsymbol{r}_k$ is the residual, $\boldsymbol{r}_k = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}_k$, and

$$\alpha_k = \frac{\boldsymbol{r}_k^T \boldsymbol{r}_k}{\boldsymbol{r}_k^T \boldsymbol{A} \boldsymbol{r}_k}.$$

1a

Compute $\boldsymbol{x}_1$ if $\boldsymbol{A} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $\boldsymbol{b} = [1\ 1]^T$ and $\boldsymbol{x}_0 = \boldsymbol{0}$.

Answer: We find $\boldsymbol{r}_0 = [1\ 1]^T$, $\alpha_0 = 1$, and $\boldsymbol{x}_1 = [1\ 1]^T$, the exact solution.

1b

If the k-th error, $\boldsymbol{e}_k = \boldsymbol{x}_k - \boldsymbol{x}$, is an eigenvector of $\boldsymbol{A}$, what can you say about $\boldsymbol{x}_{k+1}$?

Answer: If $\boldsymbol{A}\boldsymbol{e}_k = \lambda \boldsymbol{e}_k$ for some $\lambda \in \mathbb{R}$ then

$$\boldsymbol{r}_k = -\boldsymbol{A}\boldsymbol{e}_k = -\lambda \boldsymbol{e}_k,$$

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and $\alpha_k = 1/\lambda$, and therefore,

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \alpha_k \boldsymbol{r}_k = \boldsymbol{x}_k - \boldsymbol{e}_k = \boldsymbol{x},$$

which is the solution.

Problem 2 Polar Decomposition

Given $n \in \mathbb{N}$ and a singular value decomposition $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$ of a square matrix $\boldsymbol{A} \in \mathbb{R}^{n,n}$, consider the matrices

$$\boldsymbol{Q} := \boldsymbol{U}\boldsymbol{V}^T, \quad \boldsymbol{P} := \boldsymbol{V}\boldsymbol{\Sigma}\boldsymbol{V}^T \tag{1}$$

of order n.

2a

Show that

$$\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{P} \tag{2}$$

and show that $\boldsymbol{Q}$ is orthonormal.

Answer: Since $\boldsymbol{V}^T\boldsymbol{V} = \boldsymbol{I}$ we find $\boldsymbol{Q}\boldsymbol{P} = \boldsymbol{U}\boldsymbol{V}^T\boldsymbol{V}\boldsymbol{\Sigma}\boldsymbol{V}^T = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T = \boldsymbol{A}$. $\boldsymbol{Q}$ is orthonormal since $\boldsymbol{U}$ and $\boldsymbol{V}^T$ are orthonormal and a product of orthonormal matrices is orthonormal.

2b

Show that $\boldsymbol{P}$ is symmetric positive semidefinite and positive definite if $\boldsymbol{A}$ is nonsingular.

Answer: For any $\boldsymbol{x} \in \mathbb{R}^n$

$$\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x} = \boldsymbol{x}^T\boldsymbol{V}\boldsymbol{\Sigma}\boldsymbol{V}^T\boldsymbol{x} = \boldsymbol{y}^T\boldsymbol{\Sigma}\boldsymbol{y} = \sum_{j=1}^{n} \sigma_j |y_j|^2,$$

where $\boldsymbol{y} = \boldsymbol{V}^T\boldsymbol{x}$. Since the singular values σ_j are nonnegative it follows that $\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x} \geq 0$.

If $\boldsymbol{A}$ is nonsingular then $\boldsymbol{\Sigma}$ is nonsingular and then $\sigma_j > 0$ for all j. But then $\sum_{j=1}^{n} \sigma_j |y_j|^2 > 0$ for all nonzero $\boldsymbol{y} \in \mathbb{R}^n$ and then, by what we just did, $\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x} > 0$ for all nonzero $\boldsymbol{x} \in \mathbb{R}^n$.

The factorization in (2) is called a **polar factorization** of $\boldsymbol{A}$.

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2c

Use the singular value decomposition of $\boldsymbol{A}$ to give a suitable definition of $\boldsymbol{B} := \sqrt{\boldsymbol{A}^T\boldsymbol{A}}$ so that $\boldsymbol{P} = \boldsymbol{B}$.

Answer: We have $\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Sigma}\boldsymbol{U}^T\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T = \boldsymbol{V}\boldsymbol{\Sigma}^2\boldsymbol{V}^T$. If we define $\boldsymbol{B} := \boldsymbol{V}\boldsymbol{\Sigma}\boldsymbol{V}^T$ then $\boldsymbol{B} = \boldsymbol{P}$ and $\boldsymbol{B}^2 = \boldsymbol{V}\boldsymbol{\Sigma}\boldsymbol{V}^T\boldsymbol{V}\boldsymbol{\Sigma}\boldsymbol{V}^T = \boldsymbol{A}^T\boldsymbol{A}$ so we can use this $\boldsymbol{B}$ to define $\sqrt{\boldsymbol{A}^T\boldsymbol{A}}$.

For the rest of this problem assume that $\boldsymbol{A}$ is nonsingular. Consider the iterative method

$$\boldsymbol{X}_{k+1} = \frac{1}{2}\big(\boldsymbol{X}_k + \boldsymbol{X}_k^{-T}\big),\ k = 0, 1, 2, \ldots \text{ with } \boldsymbol{X}_0 = \boldsymbol{A}, \tag{3}$$

for finding $\boldsymbol{Q}$.

2d

Show that the iteration (3) is well defined by showing that $\boldsymbol{X}_k = \boldsymbol{U}\boldsymbol{\Sigma}_k\boldsymbol{V}^T$, for a diagonal matrix Σ_k with positive diagonal elements, $k = 0, 1, 2, \ldots$.

Answer: We have $\boldsymbol{X}_0 = \boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}_0\boldsymbol{V}^T$, where $\boldsymbol{\Sigma}_0 = \boldsymbol{\Sigma}$ is a diagonal matrix with positive diagonal elements. Suppose by induction that $\boldsymbol{X}_k = \boldsymbol{U}\boldsymbol{\Sigma}_k\boldsymbol{V}^T$ for a diagonal matrix $\boldsymbol{\Sigma}_k$ with positive diagonal elements. Then

$$\boldsymbol{X}_{k+1} = \frac{1}{2}\big(\boldsymbol{X}_k + \boldsymbol{X}_k^{-T}\big) = \frac{1}{2}\boldsymbol{U}\boldsymbol{\Sigma}_{k+1}\boldsymbol{V}^T,$$

where $\boldsymbol{\Sigma}_{k+1} = \frac{1}{2}\big(\Sigma_k + \Sigma_k^{-1}\big)$ is a diagonal matrix, and each diagonal element in $\boldsymbol{\Sigma}_{k+1}$ is a sum of two positive numbers and therefore positive. In particular, if $\boldsymbol{X}_k$ is nonsingular then $\boldsymbol{X}_{k+1}$ will also be nonsingular. It follows that the iteration is well defined.

2e

Show that

$$\boldsymbol{X}_{k+1} - \boldsymbol{Q} = \frac{1}{2}\boldsymbol{X}_k^{-T}\big(\boldsymbol{X}_k^T - \boldsymbol{Q}^T\big)\big(\boldsymbol{X}_k - \boldsymbol{Q}\big) \tag{4}$$

and use (4) and the Frobenius norm to show (quadratic convergence to $\boldsymbol{Q}$)

$$\|\boldsymbol{X}_{k+1} - \boldsymbol{Q}\|_F \le \frac{1}{2}\|\boldsymbol{X}_k^{-1}\|_F\|\boldsymbol{X}_k - \boldsymbol{Q}\|_F^2. \tag{5}$$

Answer: Since $\boldsymbol{X}_k = \boldsymbol{U}\boldsymbol{\Sigma}_k\boldsymbol{V}^T$ we have $\boldsymbol{X}_k^{-1} = \boldsymbol{V}\boldsymbol{\Sigma}_k^{-1}\boldsymbol{U}^T$ and $\boldsymbol{X}_k^{-T} = \boldsymbol{U}\boldsymbol{\Sigma}_k^{-1}\boldsymbol{V}^T$. But then

$$\boldsymbol{X}_k^{-T}\boldsymbol{Q}^T\boldsymbol{X}_k = \boldsymbol{U}\boldsymbol{\Sigma}_k^{-1}\boldsymbol{V}^T\boldsymbol{V}\boldsymbol{U}^T\boldsymbol{U}\Sigma_k\boldsymbol{V}^T = \boldsymbol{Q}.$$

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Expanding the right hand side of (4) and using this we find

$$\begin{aligned}
&\frac{1}{2}\boldsymbol{X}_k^{-T}(\boldsymbol{X}_k^T-\boldsymbol{Q}^T)(\boldsymbol{X}_k-\boldsymbol{Q})\\
&=\frac{1}{2}(\boldsymbol{I}-\boldsymbol{X}_k^{-T}\boldsymbol{Q}^T)(\boldsymbol{X}_k-\boldsymbol{Q})\\
&=\frac{1}{2}(\boldsymbol{X}_k-\boldsymbol{X}_k^{-T}\boldsymbol{Q}^T\boldsymbol{X}_k-\boldsymbol{Q}+\boldsymbol{X}_k^{-T}\boldsymbol{Q}^T\boldsymbol{Q})\\
&=\frac{1}{2}(\boldsymbol{X}_k+\boldsymbol{X}_k^{-T})-\boldsymbol{Q}\\
&=\boldsymbol{X}_{k+1}-\boldsymbol{Q}.
\end{aligned}$$

This shows (4).

We have $\|\boldsymbol{B}^T\|_F=\|\boldsymbol{B}\|_F$ for any matrix $\boldsymbol{B}$. Moreover, the Frobenius norm is consistent on $\mathbb{R}^{n,n}$. Therefore

$$\begin{aligned}
\|\boldsymbol{X}_{k+1}-\boldsymbol{Q}\|_F&=\|\frac{1}{2}\boldsymbol{X}_k^{-T}(\boldsymbol{X}_k^T-\boldsymbol{Q}^T)(\boldsymbol{X}_k-\boldsymbol{Q})\|_F\\
&\le\frac{1}{2}\|\boldsymbol{X}_k^{-T}\|_F\|\boldsymbol{X}_k^T-\boldsymbol{Q}^T\|_F\|\boldsymbol{X}_k-\boldsymbol{Q}\|_F\\
&=\frac{1}{2}\|\boldsymbol{X}_k^{-1}\|_F\|\boldsymbol{X}_k-\boldsymbol{Q}\|_F\|\boldsymbol{X}_k-\boldsymbol{Q}\|_F
\end{aligned}$$

and (5) follows.

2f

Write a Matlab program `function [Q,P,k] = polardecomp(A,tol,K)` to carry out the iteration in (3). The output is approximations $\boldsymbol{Q}$ and $\boldsymbol{P}=\boldsymbol{Q}^T\boldsymbol{A}$ to the polar decomposition $\boldsymbol{A}=\boldsymbol{Q}\boldsymbol{P}$ of $\boldsymbol{A}$ and the number of iterations k such that $\|\boldsymbol{X}_{k+1}-\boldsymbol{X}_k\|_F<tol*\|\boldsymbol{X}_{k+1}\|_F$. Set $k=K+1$ if convergence is not achieved in K iterations. The Frobenius norm in Matlab is written `norm(A,'fro')`.

Answer:

```
function [Q,P,k] = polardecomp(A,tol,K)
X=A;
for k=1:K
   Q=(X+inv(X'))/2;
   if norm(Q-X,'fro')<tol*norm(Q,'fro')
      P=Q'*A; return
   end
   X=Q;
end
P=[];k=K+1;
```

Good luck!