UNIVERSITETET I OSLO

Det matematisk-naturvitenskapelige fakultet

Examination in:	MAT-INF4300 — Partial differential equations and Sobolev spaces I.
Day of examination:	December 10. 2009.
Examination hours:	14.30 – 17.30.
This examination set consists of 4 pages.	
Appendices:	None.
Permitted aids:	Approved calculator.

Make sure that your copy of the examination set is complete before you start solving the problems.

Problem 1.

a. If u and v are in $H^1(\mathbb{R})$, show that

$$\int_{\mathbb{R}} uv'\,dx = -\int_{\mathbb{R}} u'v\,dx.$$

Answer: If $u \in C_c^\infty(\mathbb{R})$, then the equality holds, since it is the definition of the weak derivative. Let $\{u_k\} \in C_c^\infty(\mathbb{R}) \cap H^1(\mathbb{R})$ be a sequence such that $u_k \to u$ in $H^1(\mathbb{R})$. In particular, $u_k \to u$ and $u_k' \to u'$ in $L^2(\mathbb{R})$. Therefore

$$\int_{\mathbb{R}} uv'\,dx = \lim_{k\to\infty}\int_{\mathbb{R}} u_k'v\,dx = -\lim_{k\to\infty}\int_{\mathbb{R}} u_k v'\,dx = -\int_{\mathbb{R}} uv'\,dx.$$

b. If u and v are in $H^1(\mathbb{R})$, show that the product uv also is in $H^1(\mathbb{R})$. (Hint: Recall that $H^1(\mathbb{R}) \subset C^{0,1/2}(\mathbb{R})$.)

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Answer: Start by noting that $H^1(\mathbb{R}) = W^{1,2}(\mathbb{R}) \subset C^{0,1/2}(\mathbb{R})$, and we have the norm inequality

$$\|u\|_{L^\infty(\mathbb{R})} \le \|u\|_{C^{0,1/2}(\mathbb{R})} \le C\,\|u\|_{H^1(\mathbb{R})}\,.$$

From **(a)** it follows that for u, v in $H^1(\mathbb{R})$, $(uv)' = u'v + v'u$. Hence

$$\begin{aligned} \|uv\|_{L^2(\mathbb{R})} &\le \|u\|_{L^\infty(\mathbb{R})}\,\|v\|_{L^2(\mathbb{R})} \le C\,\|u\|_{H^1(\mathbb{R})}\,\|v\|_{H^1(\mathbb{R})} \\ \|(uv)'\|_{L^2(\mathbb{R})} &\le \|u\|_{L^\infty(\mathbb{R})}\,\|v'\|_{L^2(\mathbb{R})} + \|v\|_{L^\infty(\mathbb{R})}\,\|u'\|_{L^2(\mathbb{R})} \le C\,\|u\|_{H^1(\mathbb{R})}\,\|v\|_{H^1(\mathbb{R})}\,. \end{aligned}$$

Adding these two we find that

$$\|uv\|_{H^1(\mathbb{R})} \le C\,\|u\|_{H^1(\mathbb{R})}\,\|v\|_{H^1(\mathbb{R})}\,.$$

Problem 2.

Assume that U is a bounded open subset of $\mathbb{R}^n$ with a C^1 boundary. Let

$$X = \left\{ u \in H^1(U) \;\middle|\; \int_U u(x)\,dx = 0 \right\}.$$

a. Show that there is a constant C such that

$$\int_U u^2\,dx \le C \int_U |Du|^2\,dx \quad \text{for all } u \in X.$$

Answer: Assume not, then we can find a sequence $\{u_k\} \subset X \subset H^1(U)$ with $\|u_k\|_{L^2(U)} = 1$, such that

$$1 \ge k\,\|Du_k\|^2_{L^2(U)}\,.$$

Hence $u_k \rightharpoonup u$ in L^2 and $Du_k \to 0$ in L^2. Thus $u = c$ for some constant c. But $u \in X$, so u is L^2-orthogonal to all constants. This a contradiction.

b. Show that there is *no* constant C such that

$$\int_{\mathbb{R}^n} u^2\,dx \le C \int_{\mathbb{R}^n} |Du|^2\,dx \quad \text{for all } u \in H^1(\mathbb{R}^n).$$

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Answer: Choose u to be a any function in $H^1(\mathbb{R}^n)$. Then set $u_\alpha(x) = u(x/\alpha)$ for $\alpha > 0$. Observe that

$$Du_\alpha(x) = \frac{1}{\alpha} Du(x/\alpha).$$

We calculate

$$\begin{aligned}
\|u_\alpha\|^2_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} u^2\left(|x|/\alpha\right)\,dx \\
&= \alpha^n \int_{\mathbb{R}^n} u^2(y)\,dy = \alpha^n \|u\|^2_{L^2(\mathbb{R}^n)} \\
\|Du_\alpha\|_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \frac{1}{\alpha^2} |Du(x/\alpha)|^2\,dx \\
&= \alpha^{n-2} \int_{\mathbb{R}^n} |Du(y)|\,dy = \alpha^{n-2} \|Du\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

So if the inequality holds, then

$$\alpha^n \le C\alpha^{n-2}, \text{ for all } \alpha > 0.$$

Sending α to infinity we see that this cannot hold.

Problem 3.

Let U and X be as the previous exercise. Consider the differential equation

$$\begin{cases} -\Delta u = f \text{ in } U, \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial U, \end{cases} \tag{1}$$

where ν is the unit normal on ∂U.

a. If $u \in C^2(U) \cap C^1(\bar{U})$, is a (classical) solution to (1) show that

$$\int_U f\,dx = 0. \tag{2}$$

Answer: Using Gauss' we find that

$$0 = \int_{\partial U} \frac{du}{d\nu}\,dS = \int_U \Delta u\,dx = \int_U f\,dx.$$

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b. Show that for each $f \in L^2(U)$ there is a unique $u \in X$ such that

$$\int_U Du \cdot Dv \, dx = \int_U fv \, dx \text{ for all } v \in X. \tag{3}$$

Answer: Let B be the bilinear form defined by the left hand side of (3), taking arguments in X. Clearly it is continuous and symmetric. We must show that it coersive to use Riesz representation theorem and conclude that there is a unique weak solution. Let $u \in X$ and C be the constant from 2**(a)**,

$$\begin{aligned} B[u,u] &= \int_U |Du|^2 \, dx \\ &\geq \frac{1}{2} \|Du\|^2_{L^2(U)} + \frac{1}{2C} \|u\|^2_{L^2(U)} \\ &\geq \frac{1}{2C} \|u\|^2_{H^1(U)}, \end{aligned}$$

since we can assume that $C > 1$ without loss of generality. Hence B is coersive.

c. If $f \in L^2(U)$, but $\int_U f dx \neq 0$, (3) still gives a solution u. Explain.

Answer: The right hand side of the weak formulation must be interpreted as a member of X^* (the dual space of X), since the test functions are in X. We can decompose $L^2(U)$ as an orthogonal sum $L^2(U) = Y \oplus X^\perp$, where $X^\perp$ consists of the constant functions. Any function in $X^\perp$ corresponds to the zero functional in X^*, and will give the unique weak solution $u = 0$. Hence adding a constant to f will give the same weak solution in X.

Alternatively, call the (linear) solution operator $\Delta^{-1} : L^2(U) \to X$. Then the above means that $\Delta^{-1}(c) = 0$ for all constant functions c.

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