# UNIVERSITETET I OSLO

## Det matematisk-naturvitenskapelige fakultet

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| Examination in: | Trial exam — Partial differential equations and Sobolev spaces I. |
| Day of examination: | November 18. 2009. |
| Examination hours: | 12.15 – 15.15. |

This examination set consists of 7 pages.

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| Appendices: | None. |
| Permitted aids: | Approved calculator. |

Make sure that your copy of the examination set is complete before you start solving the problems.

## Problem 1.

**a**. If $U$ is a domain, and $u \in C^\infty(U)$ and $v \in H^2(U)$, show that

$$(uv)_{x_j x_i} = u_{x_i x_j} v + u_{x_i} v_{x_j} + u_{x_j} v_{x_i} + u v_{x_i x_j}.$$

**Answer:** Let $\varphi$ be a test function, we compute

$$\begin{aligned}
&\int_U \varphi \left(u_{x_i x_j} v + u_{x_i} v_{x_j} + u_{x_j} v_{x_i} + u v_{x_i x_j}\right)\,dx \\
&\quad = \int_U \varphi u_{x_i x_j} v - (u_{x_i}\varphi)_{x_j} v - \left(u_{x_j}\varphi\right)_{x_i} v + (u\varphi)_{x_i x_j} v\,dx \\
&\quad = \int_U v\big(u_{x_i x_j}\varphi - u_{x_i x_j}\varphi - u_{x_i}\varphi_{x_j} - u_{x_i x_j}\varphi - u_{x_j}\varphi_{x_i} + u_{x_i x_j}\varphi \\
&\qquad\qquad + u_{x_i}\varphi_{x_j} + u_{x_j}\varphi_{x_i} + u\varphi_{x_i x_j}\big)\,dx \\
&\quad = \int_U v u \varphi_{x_i x_J}\,dx = \int_U (uv)_{x_i x_j}\varphi\,dx
\end{aligned}$$

(Continued on page 2.)

For the remainder of this problem, let $U$ be a bounded open subset of $\mathbb{R}^3$, with a $C^1$ boundary. You may find the theorem in Figure 1 useful.

**THEOREM 6** (General Sobolev inequalities). *Let $U$ be a bounded open subset of $\mathbb{R}^n$, with a $C^1$ boundary. Assume $u \in W^{k,p}(U)$.*

(i) *If*

$$k < \frac{n}{p}, \tag{29}$$

*then $u \in L^q(U)$, where*

$$\frac{1}{q} = \frac{1}{p} - \frac{k}{n}.$$

*We have in addition the estimate*

$$\|u\|_{L^q(U)} \le C\|u\|_{W^{k,p}(U)}, \tag{30}$$

*the constant $C$ depending only on $k, p, n$ and $U$.*

(ii) *If*

$$k > \frac{n}{p}, \tag{31}$$

*then $u \in C^{k-\left[\frac{n}{p}\right]-1,\gamma}(\bar{U})$, where*

$$\gamma = \begin{cases} \left[\frac{n}{p}\right] + 1 - \frac{n}{p}, & \text{if } \frac{n}{p} \text{ is not an integer} \\ \text{any positive number} < 1, & \text{if } \frac{n}{p} \text{ is an integer.} \end{cases}$$

*We have in addition the estimate*

$$\|u\|_{C^{k-\left[\frac{n}{p}\right]-1,\gamma}(\bar{U})} \le C\|u\|_{W^{k,p}(U)}, \tag{32}$$

*the constant $C$ depending only on $k, p, n, \gamma$ and $U$.*

Figure 1: Theorem 5.6.6. from Evans: "Partial differential equations".

**b**. Show that for $u$ and $v$ in $H^2(U)$

$$\left\|u_{x_i x_j} v\right\|_{L^2(U)} \le C \|v\|_{H^2(U)} \left\|u_{x_i x_j}\right\|_{L^2(U)}$$

where $C$ is some constant depending on $U$ only.

**Answer:** We have $U \subset \mathbb{R}^3$ and $u$, $v \in W^{2,2}(U)$. Thus in the terminology of the theorem; $p = 2$, $k = 2$ and $n = 3$, i.e., we are in case (ii) since $2 > 3/2$. Hence $u$ (and $v$) are in $C^{0,1/2}(U)$, and we have the estimate

$$\|u\|_{L^\infty(U)} \le \|u\|_{C^{0,1/2}(U)} \le C \|u\|_{H^2(U)} \,.$$

Thus

$$\int_U \left|u v_{x_i x_j}\right|^2 dx \le \|u\|^2_{L^\infty(U)} \int_U \left|v_{x_i x_j}\right|^2 dx.$$

From these two estimates follows that

$$\left\|u v_{x_i x_j}\right\|_{L^2(U)} \le \|u\|_{L^\infty(U)} \left\|v_{x_i x_j}\right\|_{L^2(U)} \le C \|u\|_{H^2(U)} \left\|v_{x_i x_j}\right\|_{L^2(U)} \,.$$

The other estimate is proved in the same way.

*(Continued on page 3.)*

**c**. Show that if $u$ is in $H^2(U)$ then $u_{x_i} \in L^6(U)$ for $i = 1, 2, 3$, and that

$$\|u_{x_i}\|_{L^p(U)} \leq C \|u\|_{H^2(U)}, \quad \text{for } 1 \leq p \leq 6.$$

**Answer:** We have that $u_{x_i} \in H^1(U) = W^{1,2}(U)$, so in the terminology of the theorem; $p = 2$, $k = 1$ and $n = 3$. Thus we are in case (i), since $1 < 3/2$. Thus $u_{x_i}$ is in $L^q(U)$, with

$$\frac{1}{q} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

By the theorem we have that $\|u_{x_i}\|_{L^6} \leq C \|u\|_{H^2}$, but since $U$ is bounded, $\|u_{x_i}\|_{L^p} \leq C \|u_{x_i}\|_{L^6}$ for $1 \leq p \leq 6$ since $U$ is bounded.

**d**. Show that if $u$ and $v$ are in $H^2(U)$, then $u_{x_i} v_{x_j}$ is in $L^2(U)$, with the estimate

$$\left\|u_{x_i} v_{x_j}\right\|_{L^2(U)} \leq C \|u\|_{H^2(U)} \|v\|_{H^2(U)}.$$

**Answer:** Using Hölder's inequality with $p = 3$, $q = 3/2$

$$\begin{aligned}
\left\|u_{x_i} v_{x_j}\right\|^2_{L^2(U)} &= \int_U \left|u_{x_i} v_{x_j}\right|^2 dx \\
&\leq \left(\int_U \left|u_{x_i}\right|^{2\cdot 3} dx\right)^{1/3} \left(\int_U \left|v_{x_j}\right|^{2\cdot 3/2} dx\right)^{2/3} \\
&= \left\|u_{x_j}\right\|^2_{L^6(U)} \|v_{x_i}\|^2_{L^3(U)} \\
&\leq C \left\|u_{x_j}\right\|^2_{L^2(U)} \|v_{x_i}\|^2_{L^2(U)}, \quad \text{since } U \text{ is bounded,} \\
&\leq C \|u\|^2_{H^2(U)} \|v\|^2_{H^2(U)}.
\end{aligned}$$

**e**. Show that if $u$ and $v$ are in $H^2(U)$, then so is the product $uv$, and we have the estimate

$$\|uv\|_{H^2(U)} \leq C \|u\|_{H^2(U)} \|v\|_{H^2(U)}.$$

Hint: use approximation of $u$ by smooth functions in $H^2(U) \cap C^\infty(U)$. We call $H^2(U)$ an *algebra* due to this property.

**Answer:** Let $\{u^n\} \in H^2(U) \cap C^\infty(U)$ such that $u^n \to u$ in $H^2(U)$. First we note that

$$\|u^n v\|^2_{L^2(U)} \leq \|u^n\|^2_{L^\infty(U)} \|v\|^2_{H^2(U)} \leq C \|u^n\|^2_{H^2(U)} \|v\|^2_{H^2(U)}.$$

*(Continued on page 4.)*

By (**b**),

$$(u^n v)_{x_i} = u^n_{x_i} v + u^n v_{x_i}$$
$$(u^n v)_{x_i x_j} = u^n_{x_i x_j} v + u^n_{x_i} v_{x_i} + u^n_{x_j} v_{x_i} + u^n v_{x_i x_j}.$$

Hence

$$\begin{aligned}\|(u^n v)_{x_i}\|_{L^2(U)} &\leq \left\|u^n_{x_i} v\right\|_{L^2(U)} + \|u^n v_{x_i}\|_{L^2(U)} \\ &\leq \left\|u^n_{x_i}\right\|_{L^2(U)} \|v\|_{L^\infty(U)} + \|u^n\|_{L^\infty(U)} \|v\|_{L^2(U)} \\ &\leq C \|u^n\|_{H^2(U)} \|v\|_{H^2(U)},\end{aligned}$$

and

$$\begin{aligned}&\left\|(u^n v)_{x_i x_j}\right\|_{L^2(U)} \\ &\quad\leq \left\|u^n_{x_i x_j} v\right\|_{L^2(U)} + \left\|u^n_{x_i} v_{x_i}\right\|_{L^2(U)} + \left\|u^n_{x_j} v_{x_i}\right\|_{L^2(U)} + \left\|u^n v_{x_i x_j}\right\|_{L^2(U)} \\ &\quad\leq \left\|u^n_{x_i x_j}\right\|_{L^2(U)} \|v\|_{L^\infty(U)} + C \|u^n\|_{H^2(U)} \|v\|_{H^2(U)} + \|u^n\|_{L^\infty(U)} \left\|v_{x_i x_j}\right\|_{L^2(U)} \\ &\quad\leq C \|u^n\|_{H^2(U)} \|v\|_{H^2(U)}.\end{aligned}$$

Hence $u^n v \in H^2(U)$ with the estimate

$$\|u^n v\|_{H^2(U)} \leq C \|u^n\|_{H^2(U)} \|v\|_{H^2(U)}.$$

Thus $\{u^n v\} \subset H^2(U)$ satisfies

$$\|u^n v - u^m v\|_{H^2(U)} \leq C \|u^n - u^m\|_{H^2(U)} \|v\|_{H^2(U)} \to 0, \text{ as } m, n \to \infty.$$

Therefore $\{u^n v\}$ is a Cauchy sequence and $u^n v \to w \in H^2(U)$. We also have that the product $uv \in L^2(U)$,

$$\|u^n v - uv\|_{L^2(U)} \leq \|v\|_{L^\infty(U)} \|u^n - u\|_{L^2(U)} \to 0,$$

so $u^n v \to uv$ in $L^2(U)$. So for any multiindex $\alpha$ and test function $\varphi$,

$$\begin{aligned}\int_U D^\alpha w \varphi \, dx &= (-1)^{|\alpha|} \int_U w D^\alpha \varphi \, dx \\ &= \lim_{n\to\infty} (-1)^{|\alpha|} \int_U u^n v D^\alpha \varphi \, dx \\ &= (-1)^{|\alpha|} \int_U uv D^\alpha \varphi \, dx \\ &= \int_U D^\alpha (uv) \varphi \, dx,\end{aligned}$$

and thus $w = uv$ in $H^2(U)$.

*(Continued on page 5.)*

# Problem 2.

**a**. For functions in $L^2(\mathbb{R}^n)$, the Fourier transform is given by

$$\hat{u}(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot y} u(x)\, dx.$$

Show that if $\Delta u \in L^2(\mathbb{R}^n)$, then

$$\widehat{\Delta u}(y) = -\left|y\right|^2 \hat{u}(y).$$

**Answer:** We have that

$$\widehat{u_{x_i x_i}}(y) = -y_i^2 \hat{u}(y).$$

The answer follows by summing over $i$.

**b**. On the space $H^m(\mathbb{R}^n)$, we use the norm

$$\left\|u\right\|_{H^m(\mathbb{R}^n)} = \left\|(1+\left|y\right|^m)\,\hat{u}\right\|_{L^2(\mathbb{R}^n)}.$$

Show that if $u \in H^{m+1}(\mathbb{R}^n)$ and $\Delta u \in H^m(\mathbb{R}^n)$ then

$$\left\|u\right\|_{H^{m+2}(\mathbb{R}^n)} \le C\left(\left\|\Delta u\right\|_{H^m(\mathbb{R}^n)} + \left\|u\right\|_{L^2(\mathbb{R}^n)}\right),$$

for some constant $C$. Hint: The following facts may be useful: i) $(a+b)^2 \le 2(a^2+b^2)$, ii) $1+\left|y\right|^{m+2} \le 1 + (1+\left|y\right|^m)\left|y\right|^2$.

**Answer:** We have that

$$1+\left|y\right|^{m+2} \le 1 + (1+\left|y\right|^m)\left|y\right|^2.$$

Using $(a+b)^2 \le 2(a^2+b^2)$ we find that

$$\left(1+\left|y\right|^{m+2}\right)^2 \le 2\left(1 + (1+\left|y\right|^m)^2\left(\left|y\right|^2\right)^2\right).$$

Therefore

$$\begin{aligned}
\left\|u\right\|^2_{H^{m+2}(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \left(1+\left|y\right|^{m+2}\right)^2 \left|\hat{u}(y)\right|^2\, dy \\
&\le 2\Big(\int_{R^n} \left|\hat{u}(y)\right|^2\, dy + \int_{\mathbb{R}^n} (1+\left|y\right|^m)^2 \left|\left|y\right|^2 \left|\hat{u}(y)\right|\right|^2\, dy\Big) \\
&= 2\left(\left\|u\right\|^2_{L^2(\mathbb{R}^n)} + \left\|\Delta u\right\|^2_{H^m(\mathbb{R}^n)}\right) \\
&\le 2\left(\left\|u\right\|_{L^2(\mathbb{R}^n)} + \left\|\Delta u\right\|_{H^m(\mathbb{R}^n)}\right)^2.
\end{aligned}$$

*(Continued on page 6.)*

**c**. Let $c(x)$ be a bounded positive function such that $0 < c_1 \leq c(x) \leq c_2 < \infty$ for all $x \in \mathbb{R}^n$. Here $c_1$ and $c_2$ are constants. Define a weak solution to the differential equation

$$-\Delta u + c(x)u = f(x), \quad x \in \mathbb{R}^n, \tag{1}$$

where $f \in L^2(\mathbb{R}^n)$. Use the Riesz representation theorem to show that there exists a unique weak solution to (1) in $H^1(\mathbb{R}^n)$.

**Answer:** The weak formulation is

$$B[u,v] := \int_{\mathbb{R}^n} u_{x_i} v_{x_i} + c(x)uv\,dx = \int_{\mathbb{R}^n} fv\,dx = (f,v)_{L^2(\mathbb{R}^n)}$$

$u$ is called a weak solution if this holds for any $v \in H^1(\mathbb{R}^n)$. We have that

$$\begin{aligned} |B[u,v]| &\leq \max\{1, c_2\} \int_{\mathbb{R}^n} \sum_i u_{x_i} v_{x_i} + uv\,dx \\ &\leq C \|u\|_{H^1(\mathbb{R}^n)} \|v\|_{H^1(\mathbb{R}^n)}, \\ B[u,u] &\geq \min\{1, c_1\} \|u\|^2_{H^1(\mathbb{R}^n)}. \end{aligned}$$

Hence $B$ is continuous and coersive. Furthermore, $B[u,v] = B[v,u]$, and therefore $B$ defines an alternative inner product on $H^1(\mathbb{R}^n)$. Consider the linear functional $v \mapsto (f,v)_{L^2(\mathbb{R}^n)}$. This is in $H^{-1}(\mathbb{R}^n)$, and Riesz representation theorem says that there exists a unique $u \in H^1(\mathbb{R}^n)$ such that $B[u,v] = (f,v)_{L^2(\mathbb{R}^n)}$.

**d**. Show that if $u$ is a weak solution in $H^1(\mathbb{R}^n)$ of (1) then $u \in H^2(\mathbb{R}^n)$ and that

$$\|u\|_{H^2(\mathbb{R}^n)} \leq C \|f\|_{L^2(\mathbb{R}^n)},$$

for some constant $C$.

**Answer:** Let $u$ be the weak solution, and consider the linear functional on $L^2(\mathbb{R}^n)$ given by

$$v \mapsto \int_{\mathbb{R}^n} fv - cuv\,dx.$$

By Riesz representation theorem, there is a unique $h \in L^2(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} vh\,dx = \int_{\mathbb{R}^n} fv - cuv\,dx,$$

*(Continued on page 7.)*

for all $v \in L^2(\mathbb{R}^n)$. Furthermore

$$\|h\|^2_{L^2(\mathbb{R}^n)} \leq \int_{\mathbb{R}^n} |fh + cuh| \; dx \leq \|h\|_{L^2(\mathbb{R}^n)} \left( \|f\|_{L^2(\mathbb{R}^n)} + c_2 \|u\|_{L^2(\mathbb{R}^n)} \right),$$

or

$$\|h\|_{L^2(\mathbb{R}^n)} \leq C \left( \|f\|_{L^2(\mathbb{R}^n)} + \|u\|_{L^2(\mathbb{R}^n)} \right)$$

Also for every $v \in C_c^\infty(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} vh \, dx = \int_{\mathbb{R}^n} fv - cuv \, dx = \int_{\mathbb{R}^n} Du \cdot Dv \, dx = -\int_{\mathbb{R}^n} u \Delta v \, dx.$$

This means that $\Delta u = h$ in the weak sense.

Next, use (**b**) with $m = 0$, this gives

$$\|u\|_{H^2(\mathbb{R}^n)} \leq C \left( \|f\|_{L^2(\mathbb{R}^n)} + 2 \|u\|_{L^2(\mathbb{R}^n)} \right),$$

since $\|\Delta u\|_{L^2(\mathbb{R}^n)} \leq C(\|f\|_{L^2(\mathbb{R}^n)} + \|u\|_{L^2(\mathbb{R}^n)})$. However,

$$\|u\|^2_{L^2(\mathbb{R}^n)} \leq CB[u,u] \leq C \|f\|_{L^2(\mathbb{R}^n)} \|u\|_{L^2(\mathbb{R}^n)},$$

so $\|u\|_{L^2(\mathbb{R}^n)} \leq C \|f\|_{L^2(\mathbb{R}^n)}$.

END