Exercises 1 — MAT-INF4300 — Fall 2015

September 1, 2015

1

Book, Section 2.5: Problem 1

2

Show that

$$v(r) = \begin{cases} b \log r + c, & n = 2 \\ \frac{b}{r^{n-2}} + c, & n \geq 3, \end{cases}$$

where b, c are constants, satisfies the ODE

$$v''(r) + \frac{n-1}{r} v' = 0.$$

3

Define $\Phi(x)$, for $x \neq 0$, by

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log |x|, & n = 2 \\ \frac{1}{n(n-2)\alpha_n} \frac{1}{|x|^{n-2}}, & n \geq 3, \end{cases} \tag{1}$$

where α_n denotes the volume of the unit ball in $\mathbb{R}^n$.

Show that

$$D\Phi(x) = -\frac{1}{n\alpha_n} \frac{x}{|x|^n}, \qquad x \neq 0,$$

and then conclude the estimate

$$|D\Phi(x)| \leq \frac{C}{|x|^{n-1}},$$

where $C > 0$ is a constant. Moreover, show that

$$\left|D^2\Phi(x)\right| \le \frac{C}{|x|^n},$$

for some constant $C > 0$.

4

Suppose $f \in C_c^2(\mathbb{R}^n)$, and let

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy, \qquad x \in \mathbb{R}^n. \tag{2}$$

Show that

$$u_{x_i x_j}(x) = \int_{\mathbb{R}^n} \Phi(y) f_{x_i x_j}(x-y)\, dy, \qquad i,j = 1,\dots,n.$$

5

Show that there is a constant $C > 0$ such that

$$\int_{B(0,\varepsilon)} |\Phi(y)|\; dy \le \begin{cases} C\varepsilon^2 \left|\log \varepsilon\right|, & n = 2, \\ C\varepsilon^2, & n \ge 3, \end{cases}$$

where $B(0,\varepsilon) = \{x \in \mathbb{R}^n : |x| < \varepsilon\} \subset \mathbb{R}^n$ is the ball centred at the origin with radius $\varepsilon > 0$, and Φ is the fundamental solution of the Laplace's equation (1).