Exercises 2 — MAT-INF4300 — Fall 2015

September 8, 2015

1

Suppose $u \in C(\mathbb{R}^n)$. Show that, for any fixed $x \in \mathbb{R}^n$,

$$\fint_{\partial B(x,r)} u(y)\, dS(y) \to u(x), \qquad \text{as } r \to 0.$$

2

Suppose $u \in C^2(\Omega)$ satisfies

$$u(x) = \fint_{\partial B(x,r)} u(y)\, dS(y)$$

for each ball $B(x,r) \subset \Omega$. Show that $\Delta u = 0$ in Ω.

3

Let Ω be a bounded open subset of $\mathbb{R}^n$. Establish the "weak maximum principle", namely that if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is harmonic in Ω, then

$$\min_{\partial\Omega} u \le u(x) \le \max_{\partial\Omega} u, \qquad x \in \Omega.$$

Provide two proofs of this result: (i) via the strong maximum principle; (ii) a direct argument showing that $u_\varepsilon := u + \varepsilon |x|^2$, $\varepsilon > 0$, cannot attain its maximum at an interior point of Ω.

4

Book, Section 2.5: Problem 5.

5

Let $\Omega \subset \mathbb{R}^n$ be open, bounded, and fix $x, y \in \mathbb{R}^n$, $x \neq y$. Set $v(z) := G(x,z)$ and $w(z) := G(y,z)$, where G is the Green's function for the region Ω. Then

$$\Delta v = 0 \text{ in } \Omega \ (z \neq x), \qquad v = 0 \text{ on } \partial\Omega.$$

$$\Delta w = 0 \text{ in } \Omega \ (z \neq y), \qquad w = 0 \text{ on } \partial\Omega.$$

With $\varepsilon > 0$ sufficiently small, apply integration-by-parts (Green's identity) to show that

$$\int_{\partial B(x,\varepsilon)} w \frac{\partial v}{\partial \nu} - v \frac{\partial w}{\partial \nu}\, dS(z) = \int_{\partial B(y,\varepsilon)} v \frac{\partial w}{\partial \nu} - w \frac{\partial v}{\partial \nu}\, dS(z),$$

where $B(x_0, r) \subset \mathbb{R}^n$ denotes the open ball with center at x_0 and radius $r > 0$, and $\partial B(x_0, r)$ denotes the boundary of this ball. Moreover, ν denotes the inward pointing unit normal on $B(x,\varepsilon) \cup B(y,\varepsilon)$.