Exercises 3 — MAT-INF4300 — Fall 2015

September 21, 2015

1

Let $B(0,r) = \{x \in \mathbb{R}^n : |x| < r\} \subset \mathbb{R}^n$ be the open ball centred at the origin with radius $r > 0$, and denote by $\partial B(0,r) = \{x \in \mathbb{R}^n : |x| = r\}$ the boundary of the ball. Show that the solution of the boundary value problem

$$\Delta u = 0 \quad \text{in } B(0,r), \qquad u = g \quad \text{on } \partial B(0,r)$$

is given by the Poisson formula

$$u(x) = \int_{\partial B(0,r)} K(x,y) g(y)\, dS(y),$$

with K being the Poisson kernel for the ball $B(0,r)$:

$$K(x,y) = \frac{r^2 - |x|^2}{n\alpha_n r} \frac{1}{|x-y|^n}.$$

2

Let $u(x,t)$ solve the heat equation $u_t = \Delta u$ in $\mathbb{R}^n \times (0,\infty)$ and make the assumption that

$$u(x,t) = \frac{1}{t^\alpha} v\left(\frac{x}{\sqrt{t}}\right),$$

where α is a constant. Show that $v = v(y)$, with $y = x/\sqrt{t}$, solves the PDE

$$\alpha v + \frac{1}{2} y \cdot Dv + \Delta v = 0.$$

3

Let $v = v(y)$ be the function from the previous exercise. Let us make the assumption that v is radial, that is,

$$v(y) = w(|y|) = w(r), \qquad y \in \mathbb{R}^n,$$

for some function $w : \mathbb{R} \mapsto \mathbb{R}$. Show that w satisfies the ODE

$$\alpha w + \frac{r}{2} w' + w'' + \frac{n-1}{r} w' = 0. \tag{1}$$

4

Setting $\alpha = n/2$, show that (1) can be written as

$$\left(\frac{r^n}{2} w + r^{n-1} w' \right)' = 0.$$

Use this, assuming that $w, w' \to 0$ as $r \to \infty$, to arrive at the following solution of (1):

$$w(r) = Ce^{-r^2/4},$$

for some constant C.

5

Define the function u by

$$u(x,t) = C \frac{1}{t^{n/2}} \exp\left(-\frac{|x|^2}{4t} \right), \qquad x \in \mathbb{R}^n,\ t > 0,$$

where $C > 0$ is a constant. Determine C such that

$$\int_{\mathbb{R}^n} u(x,t)\, dx = 1, \qquad t > 0.$$