Mandatory assignment — MAT-INF4300 — Fall 2015

Information: The assignment must be submitted before 14:30 on Thursday, October 22, 2015, at the reception of the Department of Mathematics, the 7th floor of Niels Henrik Abels hus, Blindern. To have a passing grade you must have satisfactory answers to at least 50% of the questions and have attempted to solve all of them.

Problem 1

a)

Suppose $u \in C^2(\mathbb{R}^n \times (0,\infty))$ solves the heat equation $u_t - \Delta u = 0$. Then show that the parabolic rescaled function

$$u_\lambda(x,t) = u(\lambda x, \lambda^2 t), \qquad \lambda > 0,$$

also solves the heat equation.

b)

Use the result from a) to show that also the function $v(x,t) = x \cdot Du(x,t) + 2tu_t(x,t)$ solves the heat equation.

c)

Let η be a convex and twice continuously differentiable function, and assume u solves the heat equation. Show that the function $v = \eta(u)$ is a subsolution, i.e., $v_t - \Delta v \leq 0$.

Problem 2

Consider the initial value (Cauchy) problem

$$\begin{aligned} &u_t - \Delta u + cu = f \quad \text{in } \mathbb{R}^n \times (0,\infty), \\ &u(x,0) = u_0(x), \quad x \in \mathbb{R}^n, \end{aligned} \tag{1}$$

where c is a constant, $f \in C^{2,1}(\mathbb{R}^n \times [0,\infty))$, and $u_0 \in C(\mathbb{R}^n)$, with f, u_0 compactly supported.

a)

Write down an explicit formula for the solution $u(x,t)$ of (1). Verify that this solution candidate belongs to $C^{2,1}(\mathbb{R}^n \times (0,\infty))$, satisfies the PDE for $x \in \mathbb{R}^n$ and $t > 0$, and satisfies the initial condition in the sense that $u(x,t) \to u_0(x_0)$ as $(x,t) \to (x_0, 0)$.

b)

Suppose $f \equiv 0$ and $u \to 0$ as $|x| \to \infty$. Then use the energy method to show that for $t > 0$,

$$\|u(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq e^{-ct} \|u_0\|_{L^2(\mathbb{R}^n)} .$$

Use this result to show that there exists at most one solution of (1) with $u \to 0$ as $|x| \to \infty$.

Problem 3

Consider the nonlinear initial-boundary value problem

$$\begin{aligned} &u_t - \Delta u = -u^3 \quad \text{in } \Omega \times (0,\infty), \\ &u = u_0 \quad \text{on } \Omega \times \{t = 0\}, \\ &u = 0 \quad \text{on } \partial\Omega \times (0,\infty), \end{aligned}$$

where Ω is a bounded open subset of $\mathbb{R}^n$ and u_0 is a given continuous function. Assume that there exist twice continuously differentiable solutions to this problem.

Use the energy method to prove that

$$\|u(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq \|u_0\|_{L^2(\mathbb{R}^n)}, \qquad t > 0.$$

Problem 4

Let Ω be a bounded open subset of $\mathbb{R}^n$. Show that the Hölder space $C^{0,\gamma}(\Omega)$, with exponent $\gamma \in (0,1]$, is a Banach space.

Problem 5

Suppose Ω is a bounded open subset of $\mathbb{R}^n$ with C^1 boundary. Fix any bounded open set V that is strictly larger than Ω. Then show that there exists a bounded linear operator

$$E : W^{1,\infty}(\Omega) \to W^{1,\infty}(\mathbb{R}^n)$$

such that for each $u \in W^{1,\infty}(\Omega)$ the following properties hold:

1. $Eu = u$ almost everywhere in Ω;

2. $\operatorname{supp}(Eu) \subset V$;

3. $\|u\|_{W^{1,\infty}(\mathbb{R}^n)} \leq C \|u\|_{W^{1,\infty}(\Omega)}$,

where the constant C does not depend on u.

Problem 6

Consider the function

$$u(x) = |x - x_0|^{-\alpha}, \qquad x \in B(x_0, 1) := \{x \in \mathbb{R}^3 : |x - x_0| < 1\}, \quad x_0 \in \mathbb{R}^3,$$

where $\alpha > 0$ is a constant. Determine the weak derivative Du of u. For which values of α do we have $u \in W^{1,2}(B(x_0, 1))$.

Problem 1

a) $(u_\lambda)_t = \lambda^2 u_t(\lambda x, \lambda^2 t)$

$\Delta u_\lambda = \lambda^2 \Delta u(\lambda x, \lambda^2 t)$

So $(u_\lambda)_t - \Delta u_\lambda$

$= \lambda^2 \left(u_t(\lambda x, \lambda^2 t) - \Delta u(\lambda x, \lambda^2 t) \right)$

$\underline{= 0}$

b) Since u_λ solves the heat equ $\forall\ \lambda > 0$,

$$\frac{d}{d\lambda}\left((u_\lambda)_t - \Delta u_\lambda\right) = 0$$

so

$$\left(\frac{d}{d\lambda} u_\lambda\right)_t - \Delta\left(\frac{d}{d\lambda} u_\lambda\right) = 0$$

Set $v = \frac{d}{d\lambda} u_\lambda \Big|_{\lambda=1}$

$$= \left(x \cdot Du(\lambda x, \lambda^2 t) + 2\lambda t\, u_t(\lambda x, \lambda^2 t)\right)\Big|_{\lambda=1}$$

$$= x \cdot Du + 2t\, u_t \, ,$$

which solves the heat eqn.

c) $u_t - \Delta u = 0 \quad | \, g'(u)$

$$g(u)_t - \Delta g(u) + g''(u)|Du|^2 = 0$$

So $v = g(u)$. Then

$$v_t - \Delta v = -g''(u)|Du|^2 \leq 0,$$

since $g(\cdot)$ convex

Problem 2

a) Set $v = e^{ct} u$. Then

$$v_t = cv + e^{ct} u_t$$

$$= cv + e^{ct}(\Delta u - cu + f)$$

$$= cv + \Delta v - cv + e^{ct} f$$

$$= \Delta v + \hat{f}, \quad \underline{\hat{f} = e^{ct} f}$$

So v solves

$$v_t - \Delta v = \hat{f}$$

Solution is (book, pages 47-51)

$$V(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)\, v_0(y)\, dy$$

$$+ \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)\, \tilde{f}(y,s)\, dy\, ds$$

and thus

$$u(x,t) = e^{-ct}\, V(x,t)$$

See book, pages 47-51 for verification of solution.

b) We start from

$$u_t - \Delta u + cu = 0$$

Multiply by u and then integrate over $B(0,r)$:

$$\int_{B(0,r)} \left(\frac{u^2}{2}\right)_t dx - \int_{B(0,r)} u \Delta u \, dx + 2c \int_{B(0,r)} \frac{u^2}{2} dx = 0$$

Note that

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$$\int_{B(0,r)} u \, \Delta u \, dx$$

$$= \underbrace{\int_{\partial B(0,r)} u \frac{\partial u}{\partial \nu} \, dS}_{\to 0 \text{ as } r \to \infty} - \int_{B(0,r)} |Du|^2 dx$$

Hence, after sending $r \to \infty$,

$$\frac{d}{dt} \int_{\mathbb{R}^n} \frac{u^2}{2} dx = -2c \int_{\mathbb{R}^n} \frac{u^2}{2} dx - \int_{\mathbb{R}^n} |Du|^2 dx$$

(8)

$$\leq 2c \int_{\mathbb{R}^n} \frac{u^2}{2}\, dx$$

and so

$$\frac{d}{dt} \|u(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq -2c \|u(\cdot,t)\|_{L^2(\mathbb{R}^n)}$$

Gronwall's ineq $\Rightarrow$

$$\|u(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq e^{-2t} \|u_0\|_{L^2(\mathbb{R}^n)}$$

Regarding uniq.,
set $w = u - \hat{u}$ with
$u, \hat{u}$ being two solut.
Then w solves

$$\begin{cases} w_t - \Delta w + c w = 0 \\ w(x,0) = 0 \end{cases}$$

and thus $\underline{\underline{w = 0}}$

Problem 3

As in Problem 2 b) we find

$$\frac{d}{dt}\int_\Omega \frac{u^2}{2}\,dx$$

$$-\overbrace{\int_{\partial\Omega} u\frac{\partial u}{\partial \nu}\,dS}^{=0} + \int_\Omega |\nabla u|^2\,dx$$

$$= -\int_\Omega u^4\,dx$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\|u(\cdot,t)\|_{L^2(\Omega)}$$

$$= -\int_\Omega |\nabla u|^2 dx - \int_\Omega u^4\,dx \leq 0$$

Problem 4

Step 1 ($\|\cdot\|_{C^{0,\gamma}}$ is a norm)

Recall

$$\|u\|_{C^{0,\gamma}} = \|u\|_C + [u]_{C^{0,\gamma}},$$

where

$$\|u\|_C = \sup_{x \in \Omega} |u(x)|$$

$$[u]_{C^{0,\gamma}} = \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|u(x)-u(y)|}{|x-y|^\gamma}$$

It is easy to verify the following properties:

(1) $\|\lambda u\|_{C^{0,\gamma}} = |\lambda| \, \|u\|_{C^{0,\gamma}}$
$\forall \lambda \in \mathbb{R}$

(2) $\|u+v\|_{C^{0,\gamma}} \leq \|u\|_{C^{0,\gamma}} + \|v\|_{C^{0,\gamma}}$
$\forall u, v \in C^{0,\gamma}$

(3) $\|u\|_{C^{0,\gamma}} = 0 \Leftrightarrow u = 0$

Step 2 ($C^{0,\gamma}$ complete)

Let $\{u_m\}_{m=1}^{\infty}$ be Cauchy in $C^{0,\gamma}$. Then, for each $x \in \bar{\Omega}$, $\{u_m(x)\}_{m=1}^{\infty}$ is Cauchy in $\mathbb{R}$, and so converges to some limit $u(x)$.

Claims:

(1) $U_n \to U$ in the Hölder norm

(2) $U \in C^{0,\gamma}$ (the limit U is Hölder cont.)

Fix $\varepsilon > 0$. Since U_m is Cauchy, there $\exists N$ s.t. $m, n \geq N \Rightarrow$

$$|U_m(x) - U_n(x)| \leq \varepsilon/2$$

uniformly in x

Sending $n \to \infty$,

$$|U_m(x) - U(x)| \leq \varepsilon/2,$$

uniformly in x, $m \geq N$.

Hence

$$\begin{cases} \|u_m - u\|_C \leq \varepsilon/2 \\ \text{for } m \geq N \end{cases}$$

Moreover, again by the Cauchy prop., for all $x, y \in \overline{\Omega}$, $x \neq y$,

$$\frac{|u_m(x) - u_m(y) - (u_n(x) - u_n(y))|}{|x-y|^\gamma} \leq \frac{\varepsilon}{2}$$

for $m, n \geq N$.

Sending $n \to \infty$,

$$\frac{|U_m(x) - U_m(y) - (U(x) - U(y))|}{|x-y|^\gamma} \leqslant \frac{\varepsilon}{2}$$

for $m \geqslant N$, $\forall x, y \in \overline{\Omega}$, $x \neq y$

Hence

$$\sup_{\substack{x, y \in \Omega \\ x \neq y}} \frac{|U_m(x) - U_m(y) - (U(x) - U(y))|}{|x-y|^\gamma} \leqslant \frac{\varepsilon}{2}$$

and so

$$\|U_m - U\|_{C^{0,\gamma}} \leqslant \varepsilon$$

for $m \geqslant N$; we have convergence

Let us show that the limit is Hölder cont.

For $m \geq N$ and $x, y \in \bar{\Omega}$, $x \neq y$,

$$\frac{|U(x) - U(y)|}{|x-y|^\gamma} - \frac{|U_m(x) - U_m(y)|}{|x-y|^\gamma} \leq \varepsilon$$

by the reverse triangle ineq. | thanks to conv.

Take $\varepsilon = 1$ and recall

$$\frac{|U_m(x) - U_m(y)|}{|x-y|^\gamma} \leq \text{Const}$$

This gives

$$\frac{|u(x)-u(y)|}{|x-y|^{\gamma}} \leq 1 + \text{const}$$

So the limit is Hölder continuous.

<u>Problem 5:</u> See book, pages 268–271

<u>Problem 6:</u> See book, page 260.