

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MATMEK-4270/9270 — Numerical methods for partial differential equations

Day of examination: This is a trial exam

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1 Function approximation

The Legendre polynomials are defined as $P_0 = 1$, $P_1 = x$ and recursively

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x), \quad x \in [-1, 1], \quad (1)$$

for integer $n > 0$. The Legendre polynomials $\{P_n\}_{n=0}^N$ form a basis for $\mathbb{P}_N([-1, 1])$, which is the space of all polynomials of order less than or equal to N over $[-1, 1]$. With the Legendre basis we can define the function space $V_N = \mathbb{P}_N([-1, 1]) = \text{span}\{P_n\}_{n=0}^N$. The squared L^2 norm of the Legendre polynomials is $|P_n|^2 = \frac{2}{2n+1}$ for $n \geq 0$.

1a

Let $u(x)$ be any real function defined on the domain $[-1, 1]$. Describe the Galerkin method for approximating u with the expansion $u_N \in V_N$. Define the $L^2([-1, 1])$ inner product that is used.

1b

Use the Galerkin method to find the approximation for $u(x) = x^3$ in V_N . Give the solution as the expansion coefficients $\{\hat{u}_j\}_{j=0}^N$ and use the smallest possible N in order to obtain an exact solution.

1c

An alternative basis for $\mathbb{P}_N$ is $\{x^n\}_{n=0}^N$. Find the best approximation to $u(x) = x^3$ in $\text{span}\{x^n\}_{n=0}^N$. Why is $\{x^n\}_{n=0}^N$ a poor basis for $\mathbb{P}_N$?

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Problem 2 The Helmholtz equation

The inhomogeneous Helmholtz equation is given as

$$u''(x) + \alpha^2 u(x) = f(x), \quad x \in (-1, 1), \quad (2)$$

where $u(x)$ is the solution, $f(x)$ a real function and α a real constant. Assume at first homogeneous Dirichlet boundary conditions $u(\pm 1) = 0$ and that $V_N = \text{span}\{\psi_j\}_{j=0}^N$ is an appropriate function space for approximations to the solution $u(x)$. The basis functions ψ_j are defined such that $\psi_j(\pm 1) = 0$ for all $j \geq 0$.

2a

Describe a Galerkin method that solves Eq. (2) for an approximation $u_N \in V_N$.

2b

Describe the linear algebra problem that solves the variational problem defined in 2a. Suggest two different specific bases $\{\psi_j(x)\}_{j=0}^N$, that both can be used to solve the described problem.

2c

Assume now that $u(-1) = a$ and $u(1) = b$, where a and b are nonzero constants. Describe how the original problem can be solved with the Galerkin method using these boundary conditions.

2d

Assume now that the boundary conditions are $u'(\pm 1) = 0$. Describe how the original problem can be solved with the Galerkin method and the basis $\{P_j(x)\}_{j=0}^N$. Assume that $\alpha \neq 0$.

Problem 3 Time-dependent diffusion equation

Consider the time-dependent diffusion equation

$$\frac{\partial u}{\partial t} = c \frac{\partial^2 u}{\partial x^2}, \quad x, t \in (0, 1) \times (0, T], \quad (3)$$

where c and T are positive constants. Equation (3) is solved with boundary conditions $u(0) = u(1) = 0$ and initial condition $u(x, 0) = I(x)$ for some real function $I(x)$.

(Continued on page 3.)

3a

Describe a Crank-Nicolson finite difference method for solving the time-dependent diffusion problem. Describe the complete solution algorithm, including initialization, boundary conditions and time-stepping. Use a uniform spatial mesh, a constant time step Δt and central, second order accurate, differentiation in space.

3b

Describe a Crank-Nicolson Galerkin method for solving the time-dependent diffusion problem. Describe the complete solution algorithm, including initialization, boundary conditions, function space and time-stepping. Use a constant time step Δt . (Note: you do not need to describe the linear algebra problem.)

3c

Show that the finite difference method described in 3a is unconditionally stable. (Hint: you may use that the eigenvalues of the second order differentiation matrix are all negative and real.)