

Suggested solution

Exam MATMEK-4270/4270, 2024

Problem 1

a) Define meshfunction

$$u_j^n = u(x_j, t_n)$$

$$x_j = j\Delta x, \quad \Delta x = 1/N$$

$$t_n = n\Delta t, \quad \Delta t = T/N_t$$

$$u^n = (u_j^n)_{j=0}^N$$

Discretization using matrix form, and

$$D^{(2)} = (d_{ij}^{(2)})_{i,j=0}^{N,N}, \quad d_{ij}^{(2)} = \begin{cases} -2, & i=j, 0 < i < N \\ 1, & i=j+1, 0 < i < N \\ 1, & i=j-1, 0 < i < N \\ 0, & i=0 \vee i=N \end{cases}$$

First and last rows of $D^{(2)}$ are set to 0.

Initialize

$$u_j^0 = I(x_j), \quad j = 0, 1, \dots, N$$

For $n=0, 1, \dots, N_t-1$ solve

$$u^{n+1} = u^n + c\Delta t D^{(2)} u^n$$

7b) If $u(0)=a$ and $u(1)=b$, then define

$$u_N^n = \tilde{u}_N^n + B(x)$$

such that $\tilde{u}_N^n(0) = \tilde{u}_N^n(1)$ and

$$u_N^n(0) = B(0) = a$$

$$u_N^n(1) = B(1) = b$$

For example,

$$B(x) = (1-x)a + xb$$

In the previous algorithm, replace

u_N^n by $\tilde{u}_N^n + B$ and solve for $\tilde{u}_N^n$.

The initialization changes:

Find $\tilde{u}_N^0 \in V_N$ such that

$$(\tilde{u}_N^0 + B - I, v) = 0, \quad \forall v \in V_N$$

The time-stepping is unchanged, just replace u_N^n by $\tilde{u}_N^n$ for all n .

The time-stepping is unchanged since B is completely eliminated from the variational form.

1a) The problem can also be solved with no modification of $D^{(2)}$:

For $n=0, 1, \dots, N_t-1$ solve

$$u^{n+1} = u^n + c\Delta t D^{(2)} u^n$$

$$\text{Set } u_0^{n+1} = u_N^{n+1} = 0$$

1b) Use $V_N = \text{span} \{ \psi_j \}_{j=0}^N$

where $\psi_j(x)$ are global basis functions satisfying

$$\psi_j(0) = \psi_j(1) = 0$$

Define

$$u_N^n(x) \approx u(x, t_n)$$

Initialize by finding $u_N^0 \in V_N$ such that

$$(u_N^0 - I(x), v) = 0, \quad \forall v \in V_N$$

For $n=0, 1, \dots, N_t-1$, find $u_N^{n+1} \in V_N$ such that

$$(u_N^{n+1} - u_N^n - c\Delta t (u_N^n)'', v) = 0, \quad \forall v \in V_N$$

1c) Use ansatz

$$u^n = g^n u^0$$

for amplification factor g .

Insert into difference equation

$$u^{n+1} = u^n + c \Delta t D^{(2)} u^0$$

(Here we cannot use the modified $D^{(2)}$, because the matrix is then singular.)

$$g^{n+1} u^0 = g^n u^0 + c \Delta t g D^{(2)} u^0$$

Use eigenvalues approach

$$D^{(2)} u^0 = \lambda u^0$$

where $\lambda \in \mathbb{R}$ are the eigenvalues and u^0 the corresponding eigenvector.

$$g^{n+1} u^0 = g^n u^0 + c \Delta t g \lambda u^0$$

Divide by g and eliminate common vector u^0

$$g = 1 + c \Delta t \lambda$$

1c) All eigenvalues are real and negative. Hence

$$g = 1 - c\Delta t |\lambda|$$

Worst case is obtained for the smallest eigenvalue $\lambda_{\min}$, which has the largest $|\lambda|$.

$$g = 1 - c\Delta t |\lambda_{\min}|$$

For stability $|g| \leq 1$. g is always smaller than 1, so the limit is

$$g \geq -1$$

$$1 - c\Delta t |\lambda_{\min}| \geq -1$$

$$\Delta t \leq \frac{2}{c|\lambda_{\min}|}$$

2a) Both i) and ii) can use basis

$$\{\psi_i\}_{i=0}^N$$

where $\psi_i(x)$ are global basis functions.

i) Define $V_N = \text{span} \{\psi_i\}_{i=0}^N$

The Galerkin method is to

Find $u_N \in V_N$ such that

$$(u_N - u, v)_{L^2(0,1)} = 0, \quad \forall v \in V_N$$

Here $(a, b)_{L^2(0,1)} = \int_0^1 ab dx$

ii) Find $u_N \in V_N$ such that

$$u_N(x_i) = u(x_i), \quad i=0, 1, \dots, N$$

for a mesh $(x_i)_{i=0}^N$ of unique nodes.

2b) Define the error $e = u_N - u$ and
 $E = (e, e)$

The least squares method is to
find $u_N \in V_N$ such that

$$\frac{\partial E}{\partial \hat{u}_j} = 0, \quad \forall j = 0, 1, \dots, N$$

Here

$$u_N = \sum_{i=0}^N \hat{u}_i \psi_i$$

We get

$$\begin{aligned} \frac{\partial E}{\partial \hat{u}_j} &= \frac{\partial}{\partial \hat{u}_j} \int_0^1 e^2 dx = \int_0^1 2e \frac{\partial e}{\partial \hat{u}_j} dx \\ &= 2 \int_0^1 e \frac{\partial (u_N - u)}{\partial \hat{u}_j} dx = 2 \int_0^1 e \psi_j dx = 0 \end{aligned}$$

We see that the error is orthogonal
to the test functions ψ_j , which
is the same as the Galerkin method

2c) Define the mappings

i) $X = \frac{\underline{x}+1}{2}$ and $\underline{x} = 2X-1$

Solve the $N+1$ equations

$$u_N(x_i) = u(x_i), \quad i=0,1,\dots,N$$

where

$$u_N(x_i) = \sum_{j=0}^N \hat{u}_j l_j(\underline{x}_i)$$

ii)
$$u'(x) \approx u'_N(x) = \sum_{j=0}^N \hat{u}_j \frac{dl_j}{d\underline{x}} \frac{d\underline{x}}{dx}$$
$$= 2 \sum_{j=0}^N \hat{u}_j l'_j(\underline{x})$$

The coefficients $\hat{u}_j$ are found in i)

2d) Define $V_N = \text{span} \{ \psi_i \}_{i=0}^N$

$$W_N = V_N \otimes V_N$$

To find $u(x,y)$ with the Galerkin method is to find $u_N \in W_N$ such that

$$(u_N - u, w)_{L^2(\Omega)} = 0, \quad \forall w \in W_N$$

The inner product is defined as

$$(a, b)_{L^2(\Omega)} = \int_0^1 \int_0^1 ab \, dx \, dy$$

$$(\text{or } \int_{\Omega} ab \, d\Omega)$$

3a) Rearrange

$$u^{n+1}(1+a\Delta t\theta) = u^n(1-a\Delta t(1-\theta))$$

$$u^{n+1} = \underbrace{\frac{1+a\Delta t\theta - a\Delta t}{1+a\Delta t\theta}}_g u^n$$

Since $u^{n+1} = gu^n$, g is the amplification factor.

$$\theta = \begin{cases} 0 & : \text{Forward Euler} \\ 0.5 & : \text{Crank-Nicolson} \\ 1 & : \text{Backward Euler} \end{cases}$$

3b) For stability $0 < g \leq 1$.

If $g < 0$, then we get oscillations, which are not allowed for exponential decay problems.

3b)

$$\Theta = 0 \Rightarrow g = 1 - a\Delta t$$

g is always smaller than or equal to 1.

For stability

$$g = 1 - a\Delta t > 0$$

$$\underline{\underline{\Delta t < \frac{1}{a}}}$$

$$\Theta = 0,5 \Rightarrow g = \frac{1 - a\Delta t \cdot 0,5}{1 + a\Delta t \cdot 0,5}$$

g always ≤ 1

For stability

$$1 - a\Delta t \cdot 0,5 > 0$$

$$\underline{\underline{\Delta t < \frac{2}{a}}}$$

$$\Theta = 1 \Rightarrow g = \frac{1}{1 + a\Delta t}$$

Unconditionally stable since $0 < g \leq 1$

3c) Rearrange using $u_e^n = u_e(t_n)$

$$(*) \quad R_n(\theta) = \frac{u_e^{n+1} - u_e^n}{\Delta t} + a\theta(u_e^{n+1} - u_e^n) + a u_e^n$$

$$\text{We have } u_e(t) = e^{-at}$$

$$u_e^{(k)}(t) = (-a)^k e^{-at}$$

$$\text{where } u_e^{(k)} = \frac{d^k u_e}{dt^k} \quad k=0,1,\dots$$

$$\text{Hence } (u_e^n)^{(k)} = (-a)^k e^{-at_n} = (-a)^k u_e^n$$

The Taylor expansion can be written as

$$u_e^{n+1} - u_e^n = \Delta t(-a)u_e^n + \frac{1}{2}\Delta t^2 a^2 u_e^n - \frac{1}{6}\Delta t^3 a^3 u_e^n + O(\Delta t^4)$$

Insert Taylor expansion into (*)

$$\begin{aligned} R_n(\theta) = & \cancel{-a} u_e^n + \frac{1}{2}\Delta t a^2 u_e^n - \frac{1}{6}\Delta t^2 a^3 u_e^n + O(\Delta t^3) \\ & + a\theta \left(\Delta t(-a)u_e^n + \frac{1}{2}\Delta t^2 a^2 u_e^n - \frac{1}{6}\Delta t^3 a^3 u_e^n + O(\Delta t^4) \right) \\ & + \cancel{a} u_e^n \end{aligned}$$

$$R_n(\theta) = u_e^n \left(\frac{1}{2}\Delta t a^2 - \Delta t a^2 \theta - \frac{1}{6}\Delta t^2 a^3 + \frac{1}{2}\Delta t^2 a^3 \theta + O(\Delta t^3) \right)$$

For $\theta=0.5$ the first 2 terms are eliminated,

$$\begin{aligned} R_n(0.5) &= u_e^n \left(-\frac{1}{6}\Delta t^2 a^3 + \frac{1}{2}\Delta t^2 a^3 \frac{1}{2} + O(\Delta t^3) \right) \\ &= u_e^n \left(\frac{1}{12}\Delta t^2 a^3 + O(\Delta t^3) \right) \end{aligned}$$
