

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MATMEK-4270/9270 — Numerical methods for partial differential equations

Day of examination: This is a trial exam from 2024

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1 Function approximation in two dimensions

Let $\mathbb{P}_N(x)$ be the space of all polynomials of order less than or equal to N , defined on the real line for the variable $x \in \mathbb{R}$. The space $\mathbb{P}_N(y)$ is defined in the same way for the variable $y \in \mathbb{R}$. Let $W_N(x, y) = \mathbb{P}_N(x) \otimes \mathbb{P}_N(y)$ be the space of all product polynomials in x and y , with $(x, y) \in \mathbb{R}^2$.

1a

Let the basis for $\mathbb{P}_2(x)$ be $\{1, x, x^2\}$ and the basis for $\mathbb{P}_2(y)$ be $\{1, y, y^2\}$. Find the tensor product basis for $W_2(x, y)$.

1b

For the function $u(x, y) = 1 - y + 3x^2y$ find the best approximation $u_N(x, y) \in W_2(x, y)$ expressed in terms of its expansion coefficients $\{\hat{u}_{ij}\}_{i,j=0}^2$.

1c

Assume now that the domain is $(x, y) \in \Omega = [-1, 1]^2$. Describe the Galerkin method for approximating any function $u(x, y) \approx u_N(x, y) \in W_N(\Omega)$. Define the $L^2(\Omega)$ inner product that is required.

1d

For the domain $x \in [-1, 1]$ (and in the same way for $y \in [-1, 1]$), another basis for $\mathbb{P}_N(x)$ is $\{P_i\}_{i=0}^N$, where $P_i(x)$ is the Legendre polynomial of order i . Approximate $u(x, y) = 1 - y + 3x^2y$ using $u_N(x, y) \in W_2(x, y)$, the Galerkin

(Continued on page 2.)

method and the Legendre basis. Give the result in terms of the expansion coefficients $\{\hat{u}_{ij}\}_{i,j=0}^2$. The three lowest order Legendre polynomials are $\{1, x, \frac{1}{2}(3x^2 - 1)\}$ and $\int_{-1}^1 P_j P_i dx = \frac{2}{2i+1} \delta_{ij}$.

Problem 2 Stationary convection-diffusion equation

The stationary convection-diffusion equation is given as

$$u''(x) + \frac{1}{\varepsilon} u'(x) = 0, \quad x \in (0, 1), \quad (1)$$

with boundary conditions $u(0) = 0$ and $u(1) = 1$.

2a

Describe a central second order finite difference scheme for solving Eq. (1). Use a uniform discretization $x_j = j\Delta x$, $j = 0, 1, \dots, N$, and the notation $u_j = u(x_j)$.

2b

A general solution to a homogeneous difference equation with constant coefficients can be written as $u_j = Cg^j$, where C is a constant and g is the constant amplification factor. Show that the finite difference scheme will lead to oscillatory solutions for $\Delta x > 2\varepsilon$.

2c

Describe a global Galerkin method for solving Eq. (1).

2d

Describe a global collocation method for solving Eq. (1).

2e

Use now the Neumann boundary conditions $u'(0) = a$ and $u'(1) = b$. Describe the finite element method for solving Eq. (1) using a mesh $(x_i)_{i=0}^N$ and local piecewise linear basis functions $\psi_j(x)$ that satisfy $\psi_j(x_i) = \delta_{ij}$. (You do not need to describe $\psi_j(x)$ in detail and you do not need to assemble the complete linear problem.)

(Continued on page 3.)

2f

Describe the finite element assembly of the stiffness matrix

$$s_{ij} = (\psi'_j, \psi'_i), \quad (i, j) \in \mathcal{I}_N^2, \quad (2)$$

where $\mathcal{I}_N = \{0, 1, \dots, N\}$. (You do not need to find the exact values of s_{ij} , just describe the finite element assembly algorithm using a dense local element stiffness matrix.)

(Continued on page 4.)