

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT-MEK4270/9270 — Numerical methods for partial differential equations

Day of examination: 6 December 2024

Examination hours: 9:00 – 13:00

This problem set consists of 3 pages.

Appendices: None

Permitted aids: Approved calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1 Time-dependent diffusion equation

The diffusion equation is given as

$$\frac{\partial u(x, t)}{\partial t} = c \frac{\partial^2 u(x, t)}{\partial x^2}, \quad x, t \in (0, 1) \times (0, T], \quad (1)$$

where c and T are positive constants. The boundary conditions are $u(0) = u(1) = 0$ and the initial condition is $u(x, 0) = I(x)$.

1a

Describe a forward Euler finite difference method for solving the problem given in Eq. (1). Describe the complete algorithm, including initialization, boundary conditions and time-stepping. Describe the discrete problem using a matrix form and the second order central differentiation matrix $D^{(2)} \in \mathbb{R}^{N+1 \times N+1}$ for the diffusion term. Describe the matrix and modifications if required.

1b

Describe a forward Euler Galerkin method for solving the problem given in Eq. (1). Describe the complete solution algorithm, including initialization, boundary conditions, function space and time-stepping. Describe also the necessary modifications if the boundary conditions are changed to $u(0) = a$ and $u(1) = b$, for nonzero constants a and b .

1c

Find the stability limit (the largest possible Δt for stability) of the finite difference method described in 1a using an eigenvalue approach. You may

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use that all eigenvalues of $D^{(2)}$ are real and negative.

Problem 2 Function approximation

2a

A real function $u(x)$ is defined on the domain $x \in [0, 1]$. Explain how this function can be approximated using

- i) a global Galerkin method.
- ii) a global collocation method.

2b

Show that if the same basis is used, then the least squares method leads to exactly the same result as the Galerkin method for the problem in 2a.

2c

For the same function $u(x)$ as in 2a, let the reference coordinate $\underline{x} \in [-1, 1]$ and explain

- i) how the function $u(x)$ can be approximated using a global collocation method with Lagrange polynomials $\{\ell_i(\underline{x})\}_{i=0}^N$ for the basis.
- ii) how the derivative function $u'(x)$ can be approximated in the same Lagrange basis.

2d

A real function $v(x, y)$ is defined on the domain $\Omega = [0, 1] \times [0, 1]$. Explain how this function can be approximated using a global Galerkin method and a tensor product function space. Define the inner product required by the Galerkin method.

Problem 3 Exponential decay

The ordinary differential equation for the exponential decay of a time dependent scalar $u(t)$ can be written as

$$u'(t) = -au(t), \quad u(0) = I, \quad t \in (0, T], \quad (2)$$

where a, I and T are positive constants. The time t is discretized using uniform intervals $\Delta t = T/N_t$, where N_t is the number of time steps. A numerical solution is approximated at each time step as $u^n = u(t_n)$ for $n = 0, 1, \dots, N_t$ and $t_n = n\Delta t$. The θ -rule for numerically solving Eq. (2) is to satisfy the initial condition u^0 and solve

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$$\frac{u^{n+1} - u^n}{\Delta t} = -a(\theta u^{n+1} + (1 - \theta)u^n), \quad (3)$$

for $n = 0, 1, \dots, N_t - 1$.

3a

Show that the amplification factor for the numerical solution in Eq. (3) is

$$g = \frac{1 + a\Delta t\theta - a\Delta t}{1 + a\Delta t\theta}. \quad (4)$$

Name the three specific methods where $\theta = 0, 0.5$ and 1 .

3b

Which conditions must g satisfy for stability? Find the stability limits for the three specific cases $\theta = 0, 0.5$ and 1 .

3c

The exponential decay problem has the exact solution $u_e(t) = e^{-at}$. The truncation error, or residual $R_n(\theta)$, can be found by inserting for the exact solution in Eq. (3)

$$R_n(\theta) = \frac{u_e(t_{n+1}) - u_e(t_n)}{\Delta t} + a(\theta u_e(t_{n+1}) + (1 - \theta)u_e(t_n)). \quad (5)$$

A Taylor expansion for $u_e(t_{n+1})$ is given as

$$u_e(t_{n+1}) = u_e(t_n) + \Delta t u'_e(t_n) + \frac{1}{2}\Delta t^2 u''_e(t_n) + \frac{1}{6}\Delta t^3 u'''_e(t_n) + \mathcal{O}(\Delta t^4). \quad (6)$$

Find an expression for the residual $R_n(\theta; \Delta t, a)$, showing that for all θ , except for $\theta = 0.5$, the truncation error is first order in Δt . Show that

$$R_n(0.5; a, \Delta t) = \frac{a^3 \Delta t^2}{12} e^{-at_n} + \mathcal{O}(\Delta t^3) \quad (7)$$