

MATHEK-4270/9270 Suggested solution
Trial exam 2024

1a)

$$W_2 = \{1, x, x^2\} \otimes \{1, y, y^2\} = \{1, y, y^2, x, xy, xy^2, x^2, x^2y, x^2y^2\}$$

1b) Approximation is exact

$$u_N(x, y) = 1 - y + 3x^2y$$

$$\hat{u}_{00} = 1, \hat{u}_{01} = -1, \hat{u}_{21} = 3$$

$$\text{Remaining } \hat{u}_{ij} = 0$$

1c) The Galerkin method is to find $u_N \in W_N$ such that

$$(u_N - u, v)_{L^2(\Omega)} = 0, \quad \forall v \in W_2$$

The $L^2(\Omega)$ inner product is

$$\begin{aligned} (a, b)_{L^2(\Omega)} &= \int_{\Omega} a b d\Omega \\ &= \int_{-1}^1 \int_{-1}^1 a b dx dy \end{aligned}$$

one is sufficient.

$$1d) \quad u(x, y) = 1 - y + 3x^2y$$

$$P_0 = 1, \quad P_1 = x, \quad P_2 = \frac{1}{2}(3x^2 - 1)$$

Note that

$$u(x, y) = P_0(x)P_0(y) + 2P_2(x)P_1(y) \quad (*)$$

Hence

$$\hat{u}_{00} = 1, \quad \hat{u}_{21} = 2, \quad \text{and remaining } 0.$$

If you do not notice (*), then compute

$$(u_N - u, v) = 0 \quad \forall v \in W_2$$

$$\left(\sum_i \sum_j \hat{u}_{ij} P_i(x) P_j(y) - u, P_k(x) P_l(y) \right) = 0, \quad (k, l) \in \{0, \dots, N\}^2$$

Use $\int_{-1}^1 P_j P_i dx = \frac{2}{2i+1} \delta_{ij}$ to get

$$\hat{u}_{kl} = \frac{2k+1}{2} \frac{2l+1}{2} (u, P_k(x) P_l(y))$$

For example:

$$\begin{aligned} \hat{u}_{00} &= \frac{1}{2} \cdot \frac{1}{2} \int_{-1}^1 \int_{-1}^1 1 - y + 3x^2y \, dy \, dx \\ &= \frac{1}{4} \int_{-1}^1 \left[y - \frac{1}{2}y^2 + \frac{3}{2}x^2y^2 \right]_{-1}^1 dx \end{aligned}$$

$$7d) \quad \hat{u}_{00} = \frac{1}{4} \int_{-1}^1 \left(1 - \frac{1}{2} + \frac{3}{2}x^2\right) - \left(-1 - \frac{1}{2} + \frac{3}{2}x^2\right) dx$$

$$= \frac{1}{4} \int_{-1}^1 2 dx = \underline{\underline{1}}$$

etc.

2a) Central discretization:

$$u_{j+1} \frac{-2u_j + u_{j-1}}{\Delta x^2} + \frac{1}{\varepsilon} \left(\frac{u_{j+1} - u_{j-1}}{2\Delta x} \right) = 0$$

Boundary conditions:

$$u_0 = 0, \quad u_N = 1$$

Let $D^{(2)}$ and $D^{(1)}$ be the derivative matrices: (both of shape $N+1 \times N+1$)

$$D^{(2)} = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ & & \ddots & \ddots \\ & & 0 & 1 & -2 \end{bmatrix}, \quad D^{(1)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ & & \ddots & \ddots \\ & & 0 & -1 & 0 \end{bmatrix}$$

2a) We get the matrix problem

$$D^{(2)} u + \frac{\Delta x}{\epsilon} D^{(1)} u = b \quad , \quad b \in \mathbb{R}^{N+1}$$

$$A u = b$$

$$\text{where } A = D^{(2)} + \frac{\Delta x}{\epsilon} D^{(1)} \in \mathbb{R}^{(N+1) \times (N+1)}$$

Ident first and last rows of A .

Set $b_N = 1$, $b_i = 0$ for $i = 0, 1, \dots, N-1$

Solve

$$u = A^{-1} b$$

2b) Insert for $u_j = cg^j$ in

$$u_{j+1} - 2u_j + u_{j-1} + \frac{\Delta x}{2\varepsilon} (u_{j+1} - u_{j-1}) = 0$$

This leads to

$$g^2 - 2g + 1 + \frac{\Delta x}{2\varepsilon} (g^2 - 1) = 0$$

$$\text{set } \alpha = \frac{\Delta x}{2\varepsilon}$$

$$(1+\alpha)g^2 - 2g + 1 - \alpha = 0$$

Solve for g :

$$g = \frac{2 \pm \sqrt{4 - 4(1+\alpha)(1-\alpha)}}{2(1+\alpha)}$$

$$g = \frac{1 \pm \sqrt{1 - (1-\alpha^2)}}{1+\alpha}$$

$$g = \frac{1 \pm \alpha}{1+\alpha}$$

Oscillatory solutions for $g < 0$, obtained for $1-\alpha < 0$

$$\Rightarrow \alpha > 1 \quad \Rightarrow \quad \frac{\Delta x}{2\varepsilon} > 1 \quad \Rightarrow \quad \underline{\underline{\Delta x > 2\varepsilon}}$$

(Otherwise $|g| \leq 0$ for all Δx .)
Not required

2c) Define $V_N = \text{span} \{ \psi_i \}_{i=0}^N$

for global basisfunctions $\psi_i(x)$, $x \in [0,1]$

Boundary conditions require

$$\psi_i(0) = \psi_i(1) = 0, \quad i=0,1,\dots,N$$

Define an approximation u_N

$$u_N = \tilde{u}_N + B(x) \quad B(x) = x$$

where $\tilde{u}_N \in V_N$.

Insert for u_N in equation:

$$((\tilde{u}_N + B)'' + \frac{1}{\epsilon}(\tilde{u}_N + B)', v) = 0$$

$$(\tilde{u}_N'' + \frac{1}{\epsilon}\tilde{u}_N', v) = -\frac{1}{\epsilon}(B', v)$$

The Galerkin method is to find $\tilde{u}_N \in V_N$ such that

$$(\tilde{u}_N'' + \frac{1}{\epsilon}\tilde{u}_N', v) = -\frac{1}{\epsilon}(B', v), \quad \forall v \in V_N$$

Set $u_N = \tilde{u}_N + B$

2d) For the collocation method
find $u_N \in V_N$ such that

$$u_N''(x_i) + \frac{1}{\varepsilon} u_N'(x_i) = 0, \quad i = 1, \dots, N-1$$

$$u_N(0) = 0$$

$$u_N(1) = 1$$

Choose a mesh $\{x_i\}_{i=0}^N$, with unique nodes and $x_0 = 0, x_N = 1$.

The linear algebra problem is not required, but it does not hurt to describe it. The above answer is sufficient.

Linear algebra problem:

Choose ψ_j such that $\psi_j(x_i) = \delta_{ij}$

$$\sum_{j=0}^N \hat{u}_j \underbrace{\left(\psi_j''(x_i) + \frac{1}{\varepsilon} \psi_j'(x_i) \right)}_{a_{ij}} = b_i, \quad i = 0, \dots, N$$

Ident A in first and last rows. $A \in \mathbb{R}^{(N+1) \times (N+1)}$

Set $b_N = 1$ and $b_i = 0$ for $i = 0, \dots, N-1$

Solve

$$\underline{\hat{u}} = \underline{A}^{-1} \underline{b}$$

$$\hat{u} = \{\hat{u}_j\}_{j=0}^N, \quad b = \{b_j\}_{j=0}^N$$

2e) Define $V_N = \text{span} \{ \psi_i \}_{i=0}^N$

Use integration by parts on the variational form to get

$$(u_N'' + \frac{1}{\varepsilon} u_N', v) = 0$$

$$-(u_N', v') + \frac{1}{\varepsilon} (u_N', v) = -[u_N' v]_{x=0}^{x=1}$$

$$(*) \quad -(u_N', v') + \frac{1}{\varepsilon} (u_N', v) = -(bv(1) - av(0))$$

The Galerkin method is to find $u_N \in V_N$ such that $(*) \quad \forall v \in V_N$.

2f) Define elements $\Omega^{(e)} \in [x_e, x_{e+1}]$, $N_e = N-1$

$$S_{ij} = \int_{\Omega} \psi_j' \psi_i' dx = \sum_{e=0}^{N_e-1} \underbrace{\int_{\Omega^{(e)}} \psi_j' \psi_i' dx}_{S_{ij}^{(e)}}$$

The element stiffness matrix $S^{(e)} \in \mathbb{R}^{(k+1) \times (k+1)}$ is sparse, because there are only 2 nonzero basisfunctions in each element.

2f) Define a local, dense matrix

$$\tilde{S}_{rs}^{(e)} = \int_{\Omega^{(e)}} \psi'_{q(e,r)} \psi'_{q(e,s)} dx \in \mathbb{R}^{2 \times 2}$$

with the local-to-global map

$$q(e,r) = 2e + r$$

For elements $e = 0, 1, \dots, N_e - 1$ do

$$S_{q(e,r)q(e,s)} += \tilde{S}_{rs}^{(e)}$$

Set $S_{ij} = 0 \quad \forall (i,j) \in (0, \dots, N)^2$
