

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT-MEK4270/9270 — Numerical methods for partial differential equations

Day of examination: 19 December 2025

Examination hours: 09:00 – 13:00

This problem set consists of 3 pages.

Appendices: None

Permitted aids: Approved calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1 Function approximation

A real function $u(x)$ is defined on the domain $x \in [-1, 1]$.

1a

Describe a global Galerkin method for approximating $u(x)$ in the polynomial space $\mathbb{P}_N = \text{span}\{x^j\}_{j=0}^N$. Define the inner product used by the method. Why is $\psi_j(x) = x^j$ a poor choice of basis function? Suggest a good basis function for the chosen method and explain why this is a good basis function.

1b

Describe a collocation method for approximating $u(x)$ in the polynomial space $L_N = \text{span}\{\ell_j(x)\}_{j=0}^N$, where $\ell_j(x)$ is the j 'th Lagrange polynomial. Describe the Lagrange polynomials $\ell_j(x)$ and define $\ell_j(x)$ at all the chosen mesh points.

1c

Find an approximation to $u'(x)$ using the global Galerkin solution from 1a. In which space does the approximation to $u'(x)$ belong?

Problem 2 Poisson's equation

Poisson's equation is given as

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$$\begin{aligned}\nabla^2 u(x, y) &= f(x, y), & (x, y) &\in \Omega \\ u(x, y) &= 0, & (x, y) &\in \partial\Omega\end{aligned}\tag{1}$$

where $u(x, y)$ is the solution, $f(x, y)$ is a real smooth function, the domain $\Omega = (-1, 1)^2$ and $\partial\Omega$ is the boundary of the domain.

2a

Describe a uniform computational mesh and the meshfunctions required to solve Poisson's equation with a finite difference method. Describe also a second order accurate finite difference stencil for Eq. (1) at a mesh point away from the boundary.

2b

Describe the problem in matrix form and show how the problem can be solved using Kronecker products, including how to implement the boundary conditions. (Hint: For three matrices A, B, C , of appropriate shape such that the triple matrix product ABC^T exists, vectorization is defined as: $\text{vec}(ABC^T) = (A \otimes C)\text{vec}(B)$.)

2c

A function space is defined as $V_N(s) = \text{span}\{\psi_i(s)\}_{i=0}^N$, for basis functions $\psi_i(s)$ and a variable $s \in [-1, 1]$ and $\psi_i(\pm 1) = 0$, $\forall i = 0, 1, \dots, N$. Describe a global Galerkin method for solving Eq. (1) using the space V_N .

Problem 3 The vibration equation

The vibration equation is given as

$$u''(x) + \omega^2 u(x) = 0, \quad x \in (0, 1],\tag{2}$$

where $\omega \in \mathbb{R}$ is a constant. Two boundary conditions are given on the left hand side of the domain: $u(0) = I$, where $I \in \mathbb{R}$, and $u'(0) = 0$.

3a

Describe a second order accurate, recursive finite difference algorithm (i.e., a marching method) for solving the problem given in Eq. (2). Describe the complete algorithm, including mesh, initialization, boundary conditions, and discretization.

3b

Find the stability limit of the numerical algorithm described in 3a.

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3c

Describe a global Galerkin method for solving the problem given in Eq. (2). Use a generic function space $V_N = \text{span}\{\psi_k(x)\}_{k=0}^N$, with basis functions $\psi_k(x)$. Describe the required properties of $\psi_k(x)$.

3d

Use Legendre polynomials $P_k(\underline{x})$, $\underline{x} \in [-1, 1]$, and find $a(k)$, $b(k)$ and $\underline{x}(x)$ such that $\psi_k(x) = P_k(\underline{x}) + a(k)P_{k+1}(\underline{x}) + b(k)P_{k+2}(\underline{x})$ is a valid basis function for the function space used to solve the vibration equation. (Hint: $P_k(\pm 1) = (\pm 1)^k$ and $P'_k(\pm 1) = (\pm 1)^{k+1} \frac{k(k+1)}{2}$)