

1a) A global Galerkin method is to find $u_N \in \mathcal{P}_N$ such that

$$(u_N - u, v) = 0 \quad \forall v \in \mathcal{P}_N$$

$\psi_j = x^j$ is a poor choice of basis function since $((x^j, x^i))_{i,j=0}^N$ is a dense matrix and ill-conditioned.

The inner product is defined as

$$(a, b) = \int_{-1}^1 ab dx$$

for two real functions $a(x), b(x)$.

A good basis function for $\mathcal{P}_N$ is the Legendre polynomials $P_j(x)$:

$$\psi_j(x) = P_j(x)$$

The Legendre polynomials are good since (P_j, P_i) is diagonal.

The basis functions are orthogonal.

(Chebyshev polynomials are also good (orthogonal), but requires a different (weighted) inner product.)

1b) The Lagrange polynomials are defined as

$$l_j(x) = \prod_{\substack{k=0 \\ k \neq j}}^N \frac{(x - x_k)}{(x_j - x_k)}$$

where

$$\{x_j\}_{j=0}^N$$

are chosen (unique) mesh points.
By definition

$$l_j(x_i) = \delta_{ij}$$

The collocation method is to find $u_N \in \mathcal{L}_N$ such that

$$u_N(x_j) = u(x_j), \quad j=0, 1, \dots, N$$

Hence

$$u_N(x) = \sum_{j=0}^N u(x_j) l_j(x)$$

$$1c) \quad u'(x) \approx u'_N(x) = \sum_{j=0}^N \hat{u}_j \psi'_j(x)$$

Here $\{\hat{u}_j\}_{j=0}^N$ was found in 1a.

The space of $u'_N(x)$ is P_{N-1}

2a) Mesh points in x-direction

$$x_i = -1 + i \Delta x, \quad \Delta x = \frac{2}{N_x}, \quad i = 0, \dots, N_x$$

Mesh points in y-direction:

$$y_j = -1 + j \Delta y, \quad \Delta y = \frac{2}{N_y}, \quad j = 0, \dots, N_y$$

Define: $\vec{X} = \{x_i\}_{i=0}^{N_x}$, $\vec{Y} = \{y_j\}_{j=0}^{N_y}$

The Cartesian mesh is

$$\vec{X} \times \vec{Y} = \{(x, y) \mid x \in \vec{X} \text{ and } y \in \vec{Y}\}$$

Mesh functions:

$$u_{ij} = u(x_i, y_j), \quad i, j \in \{0, \dots, N_x\} \times \{0, \dots, N_y\}$$

$$f_{ij} = f(x_i, y_j), \quad i, j \in \{0, \dots, N_x\} \times \{0, \dots, N_y\}$$

2a) The stencil is:

$$\frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{\Delta y^2} = f_{ij}$$

2b)

Define $U = (u_{ij})_{\substack{i,j=0 \\ i,j=N_x, N_y}}$

$\bar{F} = (f_{ij})_{\substack{i,j=0 \\ i,j=N_x, N_y}}$

$$D^{(2)} = \begin{pmatrix} 2 & -5 & 4 & -1 & 0 \\ & \backslash & \backslash & \backslash & 0 \\ & & 1 & -2 & 1 \\ 0 & & & \backslash & \backslash \\ 0 & -1 & 4 & -5 & 2 \end{pmatrix} \in \mathbb{R}^{N_x+1 \times N_y+1}$$

← Not relevant, since we use explicit specification of boundary cond.

$$D_x^{(2)} = \frac{1}{\Delta x^2} D^{(2)}, \quad D_y^{(2)} = \frac{1}{\Delta y^2} D^{(2)}$$

Matrix form of equation:

$$D_x^{(2)} U + U (D_y^{(2)})^T = \bar{F}$$

2b) solve using vectorization

$$D_x^{(2)} U I_{N_y} + I_{N_x} U (D_y^{(2)})^T = F$$

where I_{N_x} and I_{N_y} are the identity matrices of shape $(N_x+1) \times (N_x+1)$ and $(N_y+1) \times (N_y+1)$, respectively.

Use vec:

$$\text{vec}(D_x^{(2)} U I_{N_y}) + \text{vec}(I_{N_x} U (D_y^{(2)})^T) = \text{vec}(F)$$

$$(D_x^{(2)} \otimes I_{N_y}) \text{vec}(U) + (I_{N_x} \otimes D_y^{(2)}) \text{vec}(U) = \text{vec}(F)$$

$$\text{Set } A = D_x^{(2)} \otimes I_{N_y} + I_{N_x} \otimes D_y^{(2)}$$

to obtain

$$A \text{vec}(U) = \text{vec}(F)$$

To enforce boundary conditions, ident all rows of A corresponding to boundary points. That is, for

all $i=0 \vee i=N_x \vee j=0 \vee j=N_y$

Set $F_{ij} = 0$ for all boundary points.

2b) Solve linear algebra problem

$$\text{vec}(U) = A^T \text{vec}(F)$$

2c) Define tensor product space

$$W_N(x, y) = V_N(x) \otimes V_N(y)$$

Find $u_N \in W_N$ such that

$$(\nabla^2 u_N - f, v) = 0, \quad \forall v \in W_N$$

Can use Green's first identity

$$-(\nabla u_N, \nabla v) - (f, v) = 0$$

but not required.

3a) Define uniform mesh

$$\vec{X} = \{x_j\}_{j=0}^N, \quad x_j = j\Delta x, \quad \Delta x = \frac{1}{N}$$

Meshfunction

$$u_j = u(j\Delta x)$$

Discretization

$$\frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2} + \omega^2 u_j = 0, \quad j = 1, \dots, N-1$$

Initialization

$$u_0 = \bar{I}$$

For second initial (boundary) condition
 $u'(0) = 0$, use 2nd order FD

$$u'(0) \approx \frac{u_1 - u_{-1}}{2\Delta x} = 0 \Rightarrow u_{-1} = u_1$$

Use ODE at $x = 0$:

$$\frac{u_1 - 2u_0 + u_{-1}}{\Delta x^2} + \omega^2 u_0 = 0$$

$$\Rightarrow u_1 = u_0 \left(1 - \frac{1}{2} \omega^2 \Delta x^2\right)$$

3a) For remaining points, march forward:

For $j = 1, 2, \dots, N-1$

$$u_{j+1} = 2u_j - u_{j-1} - \omega^2 \Delta x^2 u_j$$

3b) Use stencil:

$$u_{j+1} = (2 - \omega^2 \Delta x^2) u_j - u_{j-1}$$

Use ansatz: $u_j = g^j u_0 = g^j I$

Insert ansatz into stencil

$$g^{j+1} I = (2 - \omega^2 \Delta x^2) g^j I - g^{j-1} I$$

Divide by $I g^{j-1}$

$$g^2 = \underbrace{(2 - \omega^2 \Delta x^2)}_{\alpha} g - 1$$

$$g^2 - \alpha g + 1 = 0$$

Solve for g using quadratic formula

$$g = \frac{\alpha \pm \sqrt{\alpha^2 - 4}}{2}$$

3b) For stability the vibration equation needs constant amplitude, or $|g| = 1$.

Assume $-2 \leq \alpha \leq 2$ such that

$$\alpha^2 - 4 \leq 0$$

we get

$$g = \frac{\alpha \pm i\sqrt{4-\alpha^2}}{2}, \quad i = \sqrt{-1}$$

$$|g| = \left| \frac{\alpha \pm i\sqrt{4-\alpha^2}}{2} \right| = \frac{\alpha^2 + 4 - \alpha^2}{4} = 1$$

$\Rightarrow$ Stability achieved for

$$-2 \leq \alpha \leq 2$$

The lower limit is the relevant one

$$-2 \leq 2 - \omega^2 \Delta x^2$$

$$\omega^2 \Delta x^2 \leq 4$$

$$\Delta x \leq \frac{2}{|\omega|}$$

3b) For $\alpha > 2$ or $\alpha < -2$ $|g| \neq 1$
and the numerical solution will
be unstable.

3c) Properties of basis functions:

$$\psi_j(0) = 0 \text{ and } \psi_j'(0) = 0$$

ψ_j are all linearly independent.

Define $u_N = \tilde{u}_N + B(x)$

Let $B(0) = I$ and $B'(0) = 0$

$$\Rightarrow B(x) = I$$

The Galerkin method is to find
 $\tilde{u}_N \in V_N$ such that

$$(u_N'' + \omega^2 u_N, v) = 0 \quad \forall v \in V_N$$

$$\Rightarrow (\tilde{u}_N'' + \omega^2(\tilde{u}_N + B), v) = 0 \quad \forall v \in V_N$$

3d)

$$\psi_k(x) = P_k(x) + a(k)P_{k+1}(x) + b(k)P_{k+2}(x)$$

We need $\psi_k(0) = 0$ and $\psi'_k(0) = 0$, $\forall k = 0, \dots, N$

The mapping is

$$\underline{x} = 2x - 1, \quad \begin{array}{l} x \in [-1, 1] \\ x \in [0, 1] \end{array}$$

$$\psi_k(0) = P_k(-1) + a(k)P_{k+1}(-1) + b(k)P_{k+2}(-1)$$

$$0 = (-1)^k + a(k)(-1)^{k+1} + b(k)(-1)^k \quad \bigg/ \quad \frac{1}{(-1)^k}$$

$$\textcircled{1} \quad 0 = 1 - a(k) + b(k)$$

$$\psi'_k(0) = P'_k(-1) + a(k)P'_{k+1}(-1) + b(k)P'_{k+2}(-1)$$

$$0 = (-1)^{k+1} \frac{k(k+1)}{2} + a(k)(-1)^k \frac{(k+1)(k+2)}{2} + b(k)(-1)^{k+1} \frac{(k+2)(k+3)}{2}$$

$$\textcircled{2} \quad 0 = k(k+1) - a(k)(k+1)(k+2) + b(k)(k+2)(k+3)$$

3d) We have 2 equations with 2 unknowns (① and ②)

Solve ① for $a(k)$

$$\textcircled{1} \quad a(k) = 1 + b(k)$$

Insert ① into ② and solve for $b(k)$.

$$\begin{aligned}\textcircled{2} \quad 0 &= k(k+1) - (1+b(k))(k+1)(k+2) + b(k)(k+2)(k+3) \\ 0 &= k(k+1) - (k+1)(k+2) + b(k)((k+2)(k+3) - (k+1)(k+2)) \\ 0 &= \cancel{k^2} + k - (\cancel{k^2} + 3k + 2) + b(k)(\cancel{k^2} + 5k + 6 - (\cancel{k^2} + 3k + 2)) \\ 0 &= -2k - 2 + b(k)(2k + 4)\end{aligned}$$

$$\underline{b(k) = \frac{k+1}{k+2}}$$

$$\text{Insert into ①} \Rightarrow a(k) = 1 + \frac{k+1}{k+2} = \frac{2k+3}{k+2}$$

Solution:

$$\underline{\underline{\psi_k(x) = p_k(x) + \frac{2k+3}{k+2} p_{k+1}(x) + \frac{k+1}{k+2} p_{k+2}(x)}}$$