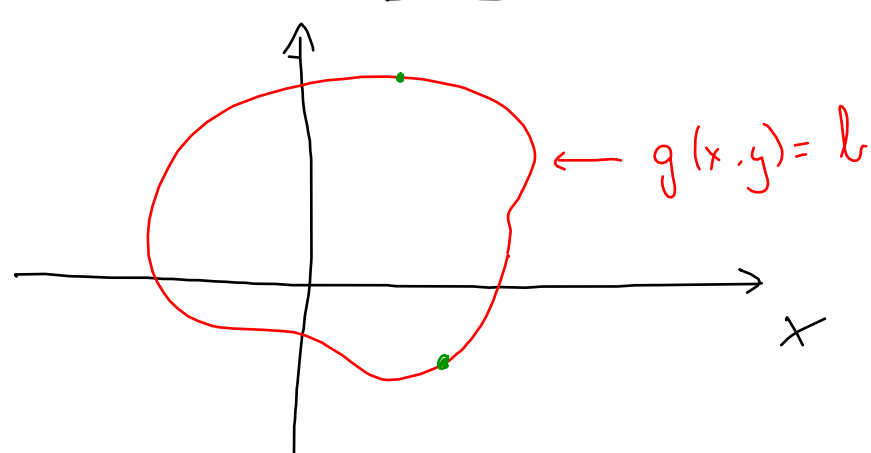


Optimering under betingelser

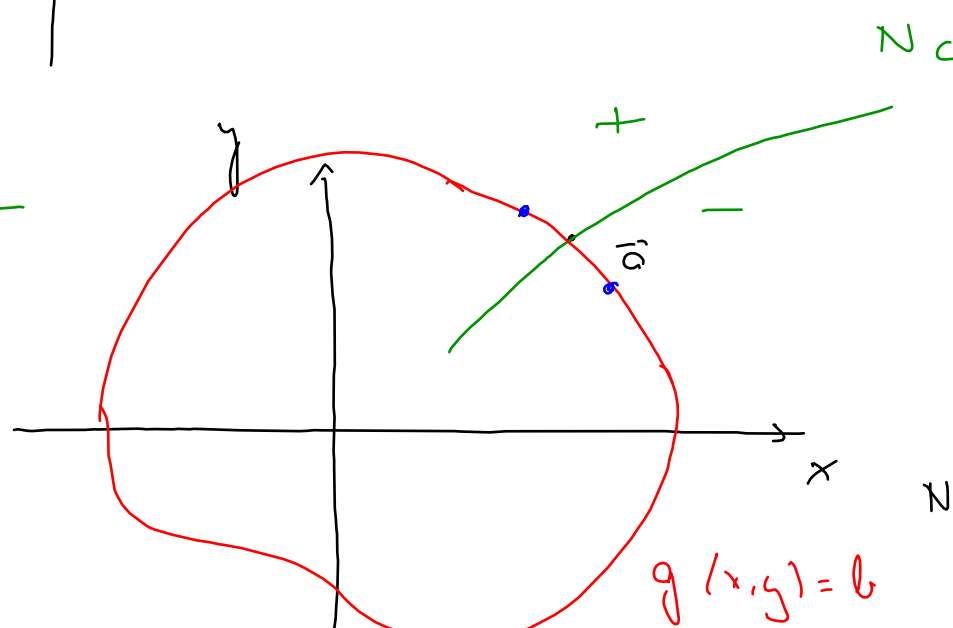
Grundlæggende problem: Find maks eller min. af

$f(x,y)$
 under betingelsen $g(x,y) = b$ or opfyldt.



Geometrisk ide

L

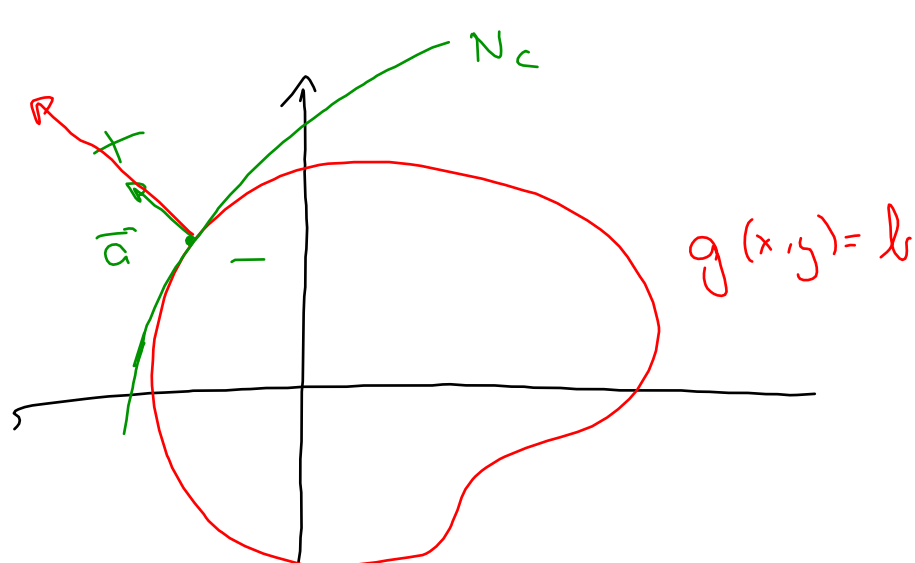


$$g(\bar{a}) = b$$

$$f(\bar{a}) = c$$

$$N_c = \{\bar{u} : f(\bar{u}) = c\}$$

Umstak



Felles tangent

$$\nabla f(\bar{a}) \parallel \nabla g(\bar{a})$$

$$\nabla f(\bar{a}) = \lambda \nabla g(\bar{a})$$

Teorem: Antag at f og g er funktioner af n -variable $(x_1, \dots, x_n)$.

Dersom f har et maksimum i $\bar{a}$ under betingelsen $g(x_1, \dots, x_n) = b$, så er enten $\nabla g(x_1, \dots, x_n) = 0$ eller det findes et λ s.d. $\nabla f(\bar{a}) = \lambda \nabla g(\bar{a})$

For at finde $\bar{a}$, må vi da løse ligningerne

$$\frac{\partial f}{\partial x_1}(x_1, \dots, x_n) = \lambda \frac{\partial g}{\partial x_1}(x_1, \dots, x_n)$$

$$\frac{\partial f}{\partial x_2}(x_1, \dots, x_n) = \lambda \frac{\partial g}{\partial x_2}(x_1, \dots, x_n)$$

$$\vdots$$

$$\frac{\partial f}{\partial x_n}(x_1, \dots, x_n) = \lambda \frac{\partial g}{\partial x_n}(x_1, \dots, x_n)$$

$$g(x_1, \dots, x_n) = b$$

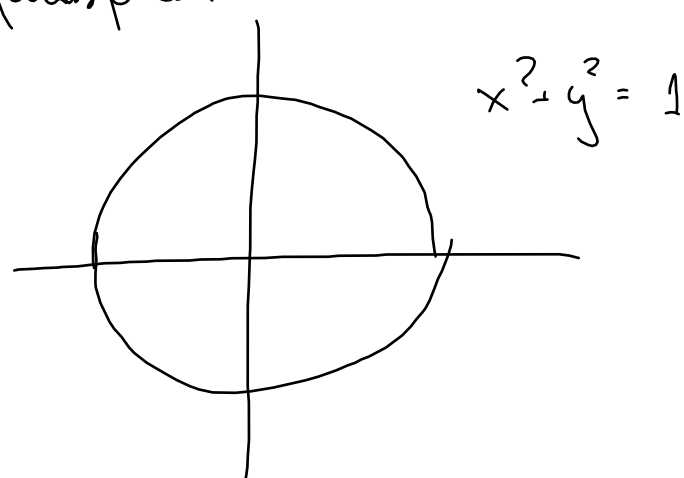
$n+1$ ligninger med
 $n+1$ ubekendte
 $x_1, \dots, x_n$ og λ

Eksempel: Find maks og min af funktionen

$$f(x,y) = xy$$

under betingelsen

$$g(x,y) = x^2 + y^2 = 1$$



Find gradienterne

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (y, x)$$

$$\nabla g(x,y) = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \right) = (2x, 2y)$$

Ligningen $\nabla f(x,y) = \lambda \nabla g(x,y)$ giver

$$(i) \quad y = \lambda 2x$$

$$(ii) \quad x = \lambda 2y \leftarrow \text{kan ikke være null}$$

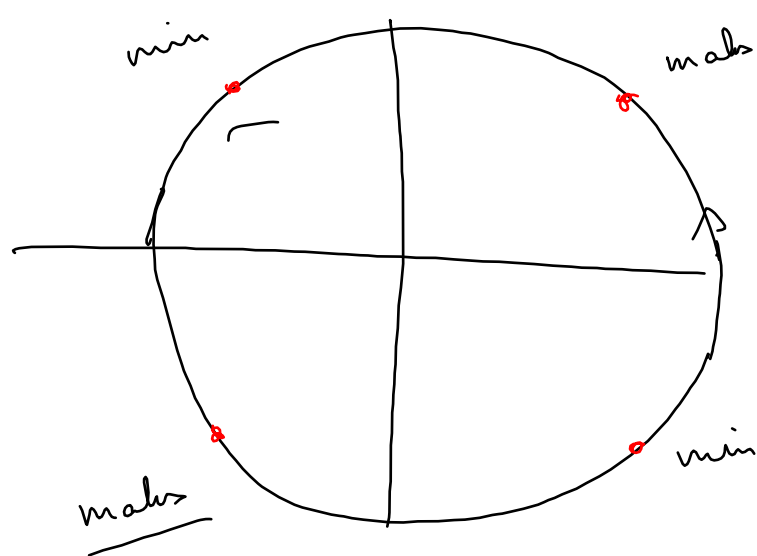
$$(iii) \quad x^2 + y^2 = 1$$

$$\text{Delen (i) på (ii)} \quad \frac{y}{x} = \frac{x}{y} \Rightarrow x^2 = y^2$$

$$\text{Sætter man i (iii)} \quad x^2 + x^2 = 1 \Rightarrow 2x^2 = 1 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm \frac{1}{\sqrt{2}} = \pm \frac{\sqrt{2}}{2}$$

$$y^2 \cdot x^2 = \frac{1}{2} \Rightarrow y = \pm \frac{\sqrt{2}}{2}$$

$$\text{Fire punkter: } \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right), \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right), \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right), \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right)$$



$$f(x,y) = xy$$

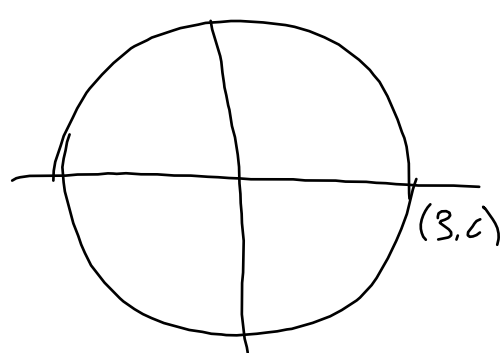
$$\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{2}{4} = \frac{1}{2}$$

Eksempel: Finn maks og min til

$$f(x, y) = x^2 + 2x + \frac{y^2}{2}$$

under betingelsen

$$g(x, y) = x^2 + y^2 = 9$$



Finner gradientene:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (2x+2, y)$$

$$\nabla g = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \right) = (2x, 2y)$$

likningen $\nabla f = \lambda \nabla g$ blir til

$$2x+2 = \lambda 2x$$

$$y = \lambda 2y \Rightarrow \begin{cases} y=0 \\ \lambda = \frac{1}{2} \end{cases}$$

$$x^2 + y^2 = 9$$

Tilfellet $y=0$: $x^2 = 9$, $x = \pm 3$ Potenzielle løsninger $(3,0), (-3,0)$

Tilfellet $\lambda = \frac{1}{2}$: $\frac{1}{2} \cdot 2x \Rightarrow 2x+2 = x \Rightarrow x+2=0 \Rightarrow \underline{x=-2}$

Dermed: $(-2)^2 + y^2 = 5 \Rightarrow y = \pm \sqrt{5}$

Potenzielle løsninger: $(-2, \sqrt{5}), (-2, -\sqrt{5})$

Sammensettingen: $f(x, y) = x^2 + 2x + \frac{y^2}{2}$

$$(3,0): f(3,0) = 3^2 + 2 \cdot 3 + \frac{0^2}{2} = 15 \leftarrow \text{maks}$$

$$(-3,0): f(-3,0) = (-3)^2 + 2(-3) + \frac{0^2}{2} = 3$$

$$(-2, \pm \sqrt{5}): f(-2, \pm \sqrt{5}) = (-2)^2 + 2(-2) + \frac{(\pm \sqrt{5})^2}{2} = 4 - 4 + \frac{5}{2} = \underline{\underline{\frac{5}{2}}}$$

$\leftarrow \text{min.}$

Oppgave 63: Ved starten av dagen x i A
 Ved slutten av dagen y i B
 z i C

$$i A : 0.7x + 0.1y + 0.1z$$

$$i B : 0.15x + 0.75y + 0.15z$$

$$i C : 0.1x + 0.1y + 0.7z$$

Setter:

$$M = \begin{bmatrix} 0.7 & 0.1 & 0.1 \\ 0.15 & 0.75 & 0.15 \\ 0.1 & 0.1 & 0.7 \end{bmatrix}$$

För opprydding: u i A
 v i B
 w i C

Etter opprydding

$$i A : 1 \cdot u + 0.2v + 0w$$

$$i B : 0u + 0.8v + 0.05w$$

$$i C : 0u + 0v + 0.95w$$

Maskinen blir

$$N = \begin{pmatrix} 1 & 0.2 & 0 \\ 0 & 0.8 & 0.05 \\ 0 & 0 & 0.95 \end{pmatrix}$$

c)

$$K = NM = \begin{pmatrix} 0.73 & 0.2 & 0.13 \\ 0.125 & 0.665 & 0.155 \\ 0.095 & 0.095 & 0.665 \end{pmatrix}$$

Ved starten av dagen $\vec{u} = \begin{pmatrix} 200 \\ 300 \\ 400 \end{pmatrix}$

] løpet av dagen

$$\vec{u} \xrightarrow{M} \vec{v} = M\vec{u} \xrightarrow{N} N\vec{v} = N(M\vec{u}) = (NM)\vec{u} = K\vec{u}$$

Fordelingen ved slutten av dagen

$$K \begin{pmatrix} 200 \\ 300 \\ 400 \end{pmatrix} = \begin{pmatrix} 255 \\ 266.5 \\ 313.5 \end{pmatrix}$$

Oppgave 74: Når har $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ en invers?

Ma undersøke når det finnes en matrise $\begin{pmatrix} x & y \\ z & u \end{pmatrix}$ slik

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & y \\ z & u \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Ganger ut

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & y \\ z & u \end{pmatrix} = \begin{pmatrix} ax+bz & ay+bu \\ cx+dz & cy+du \end{pmatrix} \stackrel{\text{Vil ha}}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Får

$$\begin{array}{l} ax+bz=1 \quad | \quad d \quad | \quad c \\ cx+dz=0 \quad | \quad -b \quad | \quad -a \end{array}$$

$$\begin{array}{l} ay+bu=0 \quad | \quad d \quad | \quad c \\ cy+du=1 \quad | \quad -b \quad | \quad -a \end{array}$$

$$adx+b dz = d$$

$$-bcx - b dz = 0$$

$$(ad-bc)x = d, \text{ hvis}$$

$$ad-bc \neq 0, \text{ får vi } x = \frac{d}{ad-bc}$$

$$ady+bdu=0$$

$$-bcy - bdu = -b$$

$$(ad-bc)y = -b \Rightarrow y = \frac{-b}{ad-bc}$$

$$acy + bdu = 0$$

$$-acy - adu = -a$$

$$-(ad-bc)z = -a$$

$$acx + bcz = c$$

$$-acx - adz = 0$$

$$(-ad+bc)z = c \Rightarrow z = \frac{-c}{ad-bc}$$

$$-(ad-bc)z = -a$$

$$u = \frac{a}{ad-bc}$$

Konklusjon: Hvis $ad-bc \neq 0$, da er

$$x = \frac{d}{ad-bc}, y = \frac{-b}{ad-bc}, z = \frac{-c}{ad-bc}, u = \frac{a}{ad-bc}$$

Inversen til $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ er

$$A^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

følgelig at $ad-bc \neq 0$.

1. a) Skjer det noen $ad-bc=0$. Da får vi ingen løsning.

b) en invers til $A = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$

$$ad-bc = 2 \cdot 3 - (-5)(-1) = 6 - 5 = 1$$

$$A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{1} \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$$

c) $\begin{array}{l} 2x - 5y = 3 \\ -x + 3y = 2 \end{array} \quad \vec{u} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$A \vec{u} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \vec{b}$$

Vi løser:

$$A \vec{u} = \vec{b} \Rightarrow A^{-1}(A \vec{u}) = A^{-1} \vec{b}$$

$$(A^{-1}A) \vec{u} = A^{-1} \vec{b}$$

$$I \vec{u} = A^{-1} \vec{b}$$

$$\vec{u} = A^{-1} \vec{b}$$

$$\vec{u} = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 9+10 \\ 3+4 \end{pmatrix} = \begin{pmatrix} 19 \\ 7 \end{pmatrix} \quad \begin{array}{l} x=19 \\ y=7 \end{array}$$