

CO₂ - rensing

Hovedrolleindehavere:

$E_{BAU}(t)$: CO₂-udslip i år t hvis "business as usual"
mål: givet (ca 7.2 i perioden 2000-2005)

$M(t)$: koncentrationen af CO₂ i atmosfæren mål i ppm
(partikler per million)

α : en omvæningsfaktor mellem E_{BAU} og M , $\alpha = 0.471$

δ : naturlig rensing $\delta = 0.01$

$M_{-\infty}$: forindenvidet koncentration

Ligning uden rensing

$$M(t+1) = M(t) + \alpha E_{BAU}(t) - \delta(M(t) - M_{-\infty})$$

$$M(t+1) = M(t) + \alpha [E_{BAU}(t)(1 - \alpha(t))] - \delta(M(t) - M_{-\infty})$$

↑
rensing.

La $\Delta M(t) = M(t) - M_{-\infty}$

$$M(t+1) - M_{-\infty} = \underbrace{M(t) - M_{-\infty}}_{\Delta M(t)} + \alpha [E_{BAU}(t)(1 - \alpha(t))] - \delta \underbrace{(M(t) - M_{-\infty})}_{\Delta M(t)}$$

$$\Delta M(t+1) = (1 - \delta) \Delta M(t) + \alpha E_{BAU}(t)(1 - \alpha(t))$$

Oppgave 3: C $n \times n$ -matrise

(i) C er positiv definit dersom

$$\vec{w}^T C \vec{w} > 0 \text{ for alle vektorer } \vec{w} \neq 0$$

(ii) C er ikke-negativ definit dersom

$$\vec{w}^T C \vec{w} \geq 0 \text{ for alle vektorer } \vec{w}.$$

a) Vis at en kovariansmatrise alltid er ikke-neg. definit.

Anta at C er kovariansmatrisen til $X_1, X_2, \dots, X_n$

$$L\alpha \quad \vec{Y} = w_1 X_1 + w_2 X_2 + \dots + w_n X_n$$

$$E[\vec{Y}] = w_1 E[X_1] + w_2 E[X_2] + \dots + w_n E[X_n] = w_1 \bar{x}_1 + w_2 \bar{x}_2 + \dots + w_n \bar{x}_n$$

Begrunn at variansen til $\vec{Y}$:

$$0 \leq E[(\vec{Y} - E[\vec{Y}])^2] = E\left[\left(\sum_{i=1}^n w_i X_i - \sum_{i=1}^n w_i \bar{x}_i\right)^2\right]$$

$$= E\left[\left(\sum_{i=1}^n w_i (X_i - \bar{x}_i)\right)^2\right] = E\left[\left(\sum_{i=1}^n w_i (X_i - \bar{x}_i)\right)\left(\sum_{j=1}^n w_j (X_j - \bar{x}_j)\right)\right]$$

$$= E\left[\sum_{i,j} w_i w_j (X_i - \bar{x}_i)(X_j - \bar{x}_j)\right] = \sum_{i,j} w_i w_j \underbrace{E[(X_i - \bar{x}_i)(X_j - \bar{x}_j)]}_{C_{ij}}$$

$$= \sum_{i,j} C_{ij} w_i w_j = \underline{\vec{w}^T C \vec{w}} \quad \text{altså er } C \text{ ikke-negativ definit!}$$

Heretter antas vi at C er positiv definit og dermed invertierbar.

b) Vis at C^{-1} er positiv definit. Anta ikke, da kan vi

finne en ikke-null vektor $\vec{u}$ slik at

$$\vec{u}^T C^{-1} \vec{u} \leq 0.$$

La $\vec{w} = C^{-1} \vec{u}$. Da er $\vec{u} \neq 0$ og $\vec{w} = C \vec{u}$. Dermed

$$0 \geq \vec{w}^T C^{-1} \vec{w} = (C \vec{u})^T C^{-1} (C \vec{u}) = \vec{u}^T C^T \vec{u} = \vec{u}^T C \vec{u}$$

Dette er en motsigelse siden C er positiv definit.

c) $F(\vec{w}) = \vec{w}^T C \vec{w} + \vec{a}^T \vec{w}$ ($\vec{a}$ er en god vektor)

Finns et punkt $\vec{w}_0$ slik at $\nabla F(\vec{w}_0) = 0$.

$$\nabla F = 2C\vec{w} + \vec{a}$$

$$\text{Setter } \nabla F = 0: 2C\vec{w} + \vec{a} = 0 \Rightarrow \underline{C\vec{w} = -\frac{1}{2}\vec{a}} \Rightarrow \underline{\vec{w} = -\frac{1}{2}C^{-1}\vec{a}}.$$

$$\text{Vi må ha } \underline{\vec{w}_0 = -\frac{1}{2}C^{-1}\vec{a}}. \quad \leftarrow C\vec{w}_0 = -\frac{1}{2}\vec{a}$$

d) Vis at $F(\vec{w}_0 + \vec{h}) = F(\vec{w}_0) + \vec{h}^T C \vec{h} + \vec{a}^T \vec{h}$ ($\vec{w}_0$ er et endelig globalt minimum)

Vi har $\vec{w}_0 + \vec{h}$

$$F(\vec{w}_0 + \vec{h}) = (\vec{w}_0 + \vec{h})^T (C(\vec{w}_0 + \vec{h}) + \vec{a}) = (\vec{w}_0^T + \vec{h}^T) (C\vec{w}_0 + C\vec{h} + \vec{a}) + \vec{a}^T \vec{w}_0 + \vec{a}^T \vec{h}$$

$$= \vec{w}_0^T C \vec{w}_0 + \vec{w}_0^T C \vec{h} + \vec{h}^T C \vec{w}_0 + \vec{h}^T C \vec{h} + \vec{a}^T \vec{w}_0 + \vec{a}^T \vec{h}$$

$$= F(\vec{w}_0) + \vec{h}^T C \vec{h} + \underbrace{\vec{w}_0^T C \vec{h} + \vec{h}^T C \vec{w}_0}_{\vec{a}^T \vec{h}} + \vec{a}^T \vec{h}$$

$$= F(\vec{w}_0) + \vec{h}^T C \vec{h} + \underbrace{(C^T \vec{w}_0)^T \vec{h}}_{C\vec{w}_0 = -\frac{1}{2}\vec{a}} + \vec{h}^T (-\frac{1}{2}\vec{a}) + \vec{a}^T \vec{h}$$

$$= F(\vec{w}_0) + \vec{h}^T C \vec{h} + \underbrace{(-\frac{1}{2}\vec{a})^T \vec{h}}_{\vec{a}^T \vec{h}} + \vec{h}^T (-\frac{1}{2}\vec{a}) + \vec{a}^T \vec{h}$$

$$= F(\vec{w}_0) + \vec{h}^T C \vec{h} - \frac{1}{2} \vec{a}^T \vec{h} - \frac{1}{2} \vec{a}^T \vec{h} + \vec{a}^T \vec{h} = F(\vec{w}_0) + \vec{h}^T C \vec{h}$$

d) Vis at $\vec{r} \vec{r}^T$ er ikke-negativt definit.

$$\vec{w}^T \vec{r} \vec{r}^T \vec{w} = (\vec{r}^T \vec{w})^T (\vec{r}^T \vec{w}) \geq 0$$

$$\vec{w}^T (C + \vec{r} \vec{r}^T) \vec{w} = \underbrace{\vec{w}^T C \vec{w}}_0 + \underbrace{\vec{w}^T \vec{r} \vec{r}^T \vec{w}}_0 > 0 \text{ for alle } \vec{w} \neq 0.$$

f)] heftet skal vi minimere

$$F(\vec{w}) = \vec{w}^T C \vec{w} + (\vec{w}^T \vec{r} - \mu_E)^2 + \text{Var}(z)$$

$$= \vec{w}^T C \vec{w} + (\vec{w}^T \vec{r})^2 - 2\mu_E \vec{w}^T \vec{r} + \mu_E^2 + \text{Var}(z)$$

$$= \vec{w}^T C \vec{w} + \vec{w}^T \vec{r} \vec{r}^T \vec{w} - 2\mu_E \vec{r}^T \vec{w} + \text{konstant}$$

$$= \vec{w}^T (C + \vec{r} \vec{r}^T) \vec{w} - \underbrace{2\mu_E \vec{r}^T \vec{w}}_{\vec{a}^T \vec{w} \text{ der } \vec{a} = -2\mu_E \vec{r}} + \text{konstant}$$

Ifølge leiem ovenfor finnes vi et endelig globalt minimum

Hvis vi setter $\nabla F = \vec{0}$.

3

Uppgave 2:

- 1 gemensamt $r_1 = 0.4$, sl. av. $\sigma_1 = 0.3$
2 " " $r_2 = 0.3$ " " $\sigma_2 = 0.25$
3 " " $r_3 = 0.5$ " " $\sigma_3 = 0.4$

$$\rho_{12} = 0.4, \rho_{13} = -0.3, \rho_{23} = -0.2$$

$$\sigma_1^2 = 0.09, \sigma_2^2 = 0.0625, \sigma_3^2 = 0.16$$

$$r_1 = 0.4$$

$$r_2 = 0.3$$

$$r_3 = 0.5$$

$$C_{12} = \rho_{12} \sigma_1 \sigma_2 = 0.4 \cdot 0.3 \cdot 0.25 = 0.03$$

$$C_{13} = \rho_{13} \sigma_1 \sigma_3 = -0.3 \cdot 0.3 \cdot 0.4 = -0.036$$

$$C_{23} = \rho_{23} \sigma_2 \sigma_3 = -0.2 \cdot 0.25 \cdot 0.4 = -0.02$$

$$C = \begin{pmatrix} 0.09 & 0.03 & -0.036 \\ 0.03 & 0.0625 & -0.02 \\ -0.036 & -0.02 & 0.16 \end{pmatrix} \quad \vec{r} \vec{r}^T = (r_i r_j) = \begin{pmatrix} 0.16 & 0.12 & 0.2 \\ 0.12 & 0.09 & 0.15 \\ 0.2 & 0.15 & 0.25 \end{pmatrix}$$

$$(C + \vec{r} \vec{r}^T) = \begin{pmatrix} 0.25 & 0.15 & 0.164 \\ 0.15 & 0.1525 & 0.13 \\ 0.164 & 0.13 & 0.41 \end{pmatrix}$$

$$(C + \vec{r} \vec{r}^T)^{-1} = \begin{pmatrix} 10.2 & -9.0 & -1.2 \\ -9.0 & 16.9 & -1.8 \\ -1.2 & -1.8 & 3.5 \end{pmatrix}$$

Maximal planering

$$(C + \vec{r} \vec{r}^T)^{-1} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} 10.2 & -9.0 & -1.2 \\ -9.0 & 16.9 & -1.8 \\ -1.2 & -1.8 & 3.5 \end{pmatrix} \begin{pmatrix} 0.4 \\ 0.3 \\ 0.5 \end{pmatrix}$$
$$= \begin{pmatrix} 0.78 \\ 0.57 \\ 0.73 \end{pmatrix}$$

Oppgave 1: $\left. \begin{array}{l} \text{Altp 1: Forenkelt utkastning } r_1 \\ \text{Altp 2: } \text{---} \text{''---} \text{ } r_2 \end{array} \right\} r_1 \neq r_2$

a) Ønsket utkastning r :

$$w_1 r_1 + w_2 r_2 = r$$

$$w_1 + w_2 = 1 \quad | -r_2$$

Løsninger: $w_1 = \frac{r - r_1}{r_1 - r_2}, w_2 = \frac{r_2 - r}{r_1 - r_2}$

b) Varianansen: $\vec{w}^T C \vec{w} = \sum_{i,j} c_{ij} w_i w_j$

$$= \overset{\sigma_1^2}{c_{11}} w_1^2 + 2 \overset{\sigma_{12}}{c_{12}} w_1 w_2 + \overset{\sigma_2^2}{c_{22}} w_2^2 \quad C = \begin{pmatrix} \sigma_1^2 & c_{12} \\ c_{21} & \sigma_2^2 \end{pmatrix}$$

$$= \sigma_1^2 \frac{(r - r_1)^2}{(r_1 - r_2)^2} + 2 c_{12} \frac{r - r_1}{r_1 - r_2} \frac{r_2 - r}{r_1 - r_2} + \sigma_2^2 \left(\frac{r_2 - r}{r_1 - r_2} \right)^2$$

c) $r_1 = 0.1, r_2 = 0.2$ $C = \frac{1}{100} \begin{pmatrix} 0.5 & -1 \\ -1 & 8 \end{pmatrix}$

Matteansvar

~~De~~ $(r_1 - r_2)^2 = (0.1 - 0.2)^2 = \frac{1}{100}$

σ_w^2 = $100 \cdot 0.5 (r - 0.1)^2 - 200 \cdot 1 (r - 0.1)(0.2 - r) + 8 \cdot 100 (r - 0.2)^2$

$$= 50 (r - 0.1)^2 + 200 (r - 0.1)(r - 0.2) + 800 (r - 0.2)^2$$

Gang ut og finn den effisiente portefølje! fronten!