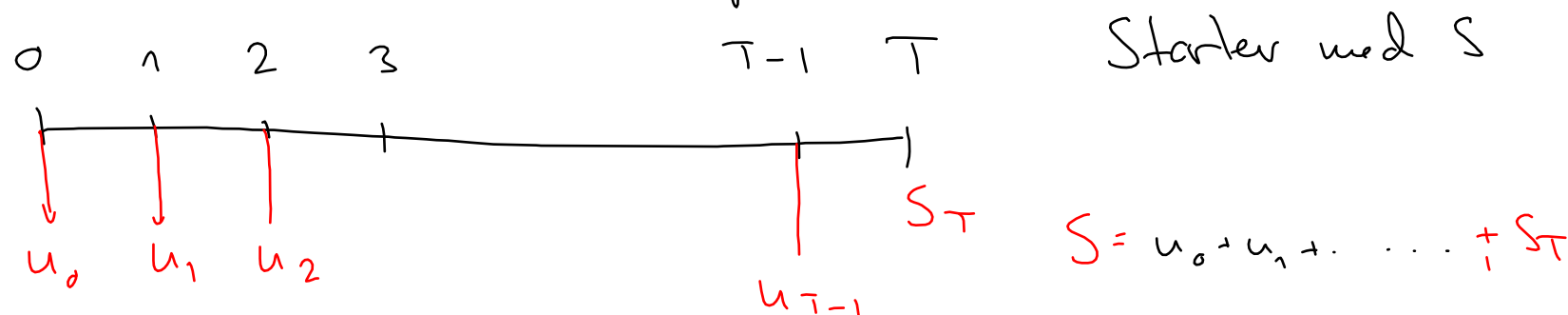


Z

Langtidsprogramering



Vil optimere

$$\sum_{t=0}^{T-1} \beta^t u_t^\eta + \beta^T S_T^\eta$$

$$0 < \beta < 1$$

typisk $\beta = 0.95$

$$\eta = 0.75$$

$$T = 20$$

Nyttig vekselsyklus: Maksimer

$$f(u) = A^{\eta-1} u^\eta + B^{\eta-1} (x-u)^\eta$$

hvor A, B og x er positive.

Deriver $f'(u) = A^{\eta-1} \eta u^{\eta-1} + B^{\eta-1} \eta (x-u)^{\eta-1} (-1)$

Sette $f'(u) = 0$ og får

$$A^{\eta-1} \cancel{\eta} u^{\eta-1} = B^{\eta-1} \cancel{\eta} (x-u)^{\eta-1}$$

Opphøyer i $\frac{1}{\eta-1}$:

$$Au = B(x-u)$$

$$(A+B)u = Bx$$

$$u^* = \frac{B}{A+B} x \quad (\text{maksimeringspunkt})$$

Maksimeringsdi:

$$f(u^*) = A^{\eta-1} u^{*\eta} + B^{\eta-1} (x-u^*)^\eta$$

$$= A^{\eta-1} \left(\frac{B}{A+B} x \right)^\eta + B^{\eta-1} \left(\frac{A}{A+B} x \right)^\eta$$

$$= \frac{x^\eta}{(A+B)^\eta} \left[A^{\eta-1} B^\eta + B^{\eta-1} A^\eta \right]$$

$$= \frac{A^{\eta-1} B^{\eta-1}}{(A+B)^\eta} x^\eta \left[B + A \right] = \frac{A^{\eta-1} B^{\eta-1}}{(A+B)^{\eta-1}} x^\eta$$

$$= \left(\frac{AB}{A+B} \right)^{\eta-1} x^\eta$$

$$\begin{aligned} x - u^* &= \frac{A+B}{A+B} x - \frac{B}{A+B} x \\ &= \frac{A}{A+B} x \end{aligned}$$

Tilblæk til approksimert represetasjon

$$\sum_{t=0}^{T-1} \beta^t u_t^\eta + \beta^T S_T^\eta$$

Introduer en hjelpeskrivare $\alpha = \beta^{\frac{1}{\eta-1}} \Rightarrow \alpha^{\eta-1} = \beta$

Dette gir at $\beta^t = (\alpha^{\eta-1})^t = (\alpha^t)^{\eta-1}$

Optimeringsprossess blir dermed

$$\sum_{t=0}^{T-1} (\alpha^t)^{\eta-1} u_t^\eta + (\alpha^T)^{\eta-1} S_T^\eta$$

Vi starter i tidspunkt $T-1$ med
ressurs x : Anta vi utvinner u , da
blir nyttet

$$(\alpha^{T-1})^{\eta-1} u^\eta + (\alpha^T)^{\eta-1} (x-u)^\eta$$

Vil velge u slik at dette uttrykket blir størst mulig

Brake hjelpeskrivaren med $A = \alpha^{T-1}$, $B = \alpha^T$

Maximumpunkt: $u_{T-1}^*(x) = \frac{B}{A+B} x = \frac{\alpha^T}{\alpha^{T-1} + \alpha^T} x$

$$= \frac{\alpha}{1+\alpha} x$$

Maximumsverdi:

$$V_{T-1}(x) = \left(\frac{AB}{A+B} \right)^{\eta-1} x^\eta = \left(\frac{\cancel{\alpha^{T-1}} \alpha^T}{\cancel{\alpha^{T-1}} + \alpha} \right)^{\eta-1} x^\eta$$

$$= \left(\frac{\alpha^T}{1+\alpha} \right)^{\eta-1} x^\eta$$

Fluor skjer ved tiden $T-2$. Anta at vi har en ressurs x
og ønsker å utvinne u :

$$\underbrace{(\alpha^{T-2})^{\eta-1}}_{\beta^{T-2}} u^\eta + V_{T-1}(x-u)$$

$$= (\alpha^{T-2})^{\eta-1} u^\eta + \left(\frac{\alpha^T}{1+\alpha} \right)^{\eta-1} (x-u)^\eta$$

Vil velge u slik at dette blir størst mulig

(forhåndsregning med $A = \alpha^{T-2}$, $B = \frac{\alpha^T}{1+\alpha}$)

Maximal u -verdi: $u_{T-2}^*(x) = \frac{B}{A+B} x = \frac{\frac{\alpha^T}{1+\alpha}}{\alpha^{T-2} + \frac{\alpha^T}{1+\alpha}}$

$$= \frac{\frac{\alpha^2}{1+\alpha}}{1 + \frac{\alpha^2}{1+\alpha}} = \frac{\alpha^2}{1+\alpha+\alpha^2}$$

Maximalt utbytte:

$$V_{T-2}(x) = \left(\frac{AB}{A+B} \right)^{\eta-1} x^\eta = \left(\frac{\cancel{\alpha^{T-2}} \frac{\alpha^T}{1+\alpha}}{\cancel{\alpha^{T-2}} + \frac{\alpha^T}{1+\alpha}} \right)^{\eta-1} x^\eta$$

$$= \left(\frac{\frac{\alpha^T}{1+\alpha}}{1 + \frac{\alpha^2}{1+\alpha}} \right)^{\eta-1} x^\eta = \left(\frac{\alpha^T}{1+\alpha+\alpha^2} \right)^{\eta-1} x^\eta$$

Hvis vi på samme måte regner ut $u_{T-3}^*(x) = \frac{\alpha^3}{1+\alpha+\alpha^2+\alpha^3}$

$$\text{og } V_{T-3}(x) = \left(\frac{\alpha^T}{1+\alpha+\alpha^2+\alpha^3} \right)^{\eta-1} x^\eta$$

Hypotese: $u_{T-k}^*(x) = \frac{\alpha^k}{1+\alpha+\alpha^2+\dots+\alpha^k} x$

$$V_{T-k}(x) = \left(\frac{\alpha^T}{1+\alpha+\alpha^2+\dots+\alpha^k} \right)^{\eta-1} x^\eta$$

Beviser hypotesen ved induksjon, dvs antar at

den gjelder for en k og viser at den da holder

for $k+1$. Anta at vi ved tidspunkt $T-(k+1)$

= $T-k-1$ har en ressurs x og at vi utvinner u

Nyttet er da

$$\beta^{T-k-1} u^\eta + V_{T-k}(x-u)$$

$$= (\alpha^{T-k-1})^{\eta-1} u^\eta + \left(\frac{\alpha^T}{1+\alpha+\alpha^2+\dots+\alpha^k} \right)^{\eta-1} (x-u)^\eta$$

Den optimale u er da (brake forhåndsregning med

$A = \alpha^{T-k-1}$, $B = \frac{\alpha^T}{1+\alpha+\alpha^2+\dots+\alpha^k}$)

$$u_{T-k-1}^*(x) = \frac{B}{A+B} x$$

$$= \frac{\alpha^T}{1+\alpha+\alpha^2+\dots+\alpha^k} \cdot \frac{1}{\alpha^{T-k-1} + \frac{\alpha^T}{1+\alpha+\alpha^2+\dots+\alpha^k}}$$

forklarer med α^{T-k-1}
 $= \alpha^{T-(k+1)}$

$$= \frac{\alpha}{1+\alpha+\dots+\alpha^k}$$

ganger med $1+\alpha+\dots+\alpha^k$

$$= \frac{1}{1 + \frac{\alpha^{k+1}}{1+\alpha+\dots+\alpha^k}}$$

oppes og ved

$$= \frac{\alpha^{k+1}}{1+\alpha+\dots+\alpha^k + \alpha^{k+1}} \quad \text{Herre!}$$

Tilsvarende blir maksimalverdien

$$A = \alpha^{T-k-1}$$

$$V_{T-(k+1)}(x) = \left(\frac{AB}{A+B} \right)^{\eta-1} x^\eta =$$

$$B = \frac{\alpha^T}{1+\alpha+\dots+\alpha^k}$$

$$= \left(\frac{\alpha^{T-k-1} \frac{\alpha^T}{1+\alpha+\dots+\alpha^k}}{\alpha^{T-k-1} + \frac{\alpha^T}{1+\alpha+\dots+\alpha^k}} \right)^{\eta-1} x^\eta$$

forklarer med α^{T-k-1}

$$= \left(\frac{\frac{\alpha^T}{1+\alpha+\dots+\alpha^k}}{1 + \frac{\alpha^{k+1}}{1+\alpha+\dots+\alpha^k}} \right)^{\eta-1} x^\eta$$

ganger med $1+\alpha+\dots+\alpha^k$ oppes og ved

$$= \left(\frac{\alpha^T}{1+\alpha+\dots+\alpha^k + \alpha^{k+1}} \right)^{\eta-1} x^\eta$$

Dette avslutter induktionsbeviset.

Oppsummering: $V_{T-k}(x) = \left(\frac{a^T}{1+a+\dots+a^k} \right)^{n-1} x^n$

$$u_{T-k}^*(x) = \frac{a^k}{1+a+\dots+a^k}$$

Summen geometrisk rekke: $1+a+\dots+a^k = \frac{a^{k+1}-1}{a-1}$

Så vi gir:

$$V_{T-k}(x) = \left(\frac{a^T}{\frac{a^{k+1}-1}{a-1}} \right)^{n-1} x^n = \left(\frac{a^T(a-1)}{a^{k+1}-1} \right)^{n-1} x^n$$

$$u_{T-k}^*(x) = \frac{a^k}{\frac{a^{k+1}-1}{a-1}} = \frac{a^k(a-1)}{a^{k+1}-1}$$

Setter $t = T-k$, dvs $k = T-t$. Da er

$$V_t(x) = \left(\frac{a^T(a-1)}{a^{T-t+1}-1} \right)^{n-1} x^n$$

$$= \left(\frac{a^t a^{T-t}(a-1)}{a^{T-t+1}-1} \right)^{n-1} x^n = (a^t)^{n-1} = \rho^t$$

$$= \left(\frac{a^{T-t}(a-1)}{a^{T-t+1}-1} \right)^{n-1} \rho^t x^n \quad \text{del med } a^{T-t}$$

$$= \left(\frac{(a-1)}{a - a^{t-T}} \right)^{n-1} \rho^t x^n$$

$$= b(t)^{n-1} \rho^t x^n$$

der $b(t) = \frac{a-1}{a - a^{t-T}}$

Tilsvarende $t = T-k$, $k = T-t$

$$u_t^*(x) = \frac{a^{T-t}(a-1)}{a^{T-t+1}-1} x \quad \text{del med } a^{T-t}$$

$$= \frac{a-1}{a - a^{t-T}} x = \underline{b(t) x}$$

$$u_{T-k}^*(x) = \frac{a^k}{1+\dots+a^k} x$$

$$= \frac{a^k}{\frac{a^{k+1}-1}{a-1}} x = \frac{a^k(a-1)}{a^{k+1}-1} x$$

Ved å bruke dette systemet, finner vi at

$$S_t = \frac{a^{T+1-t}-1}{a^{T+1}-1} S_0$$