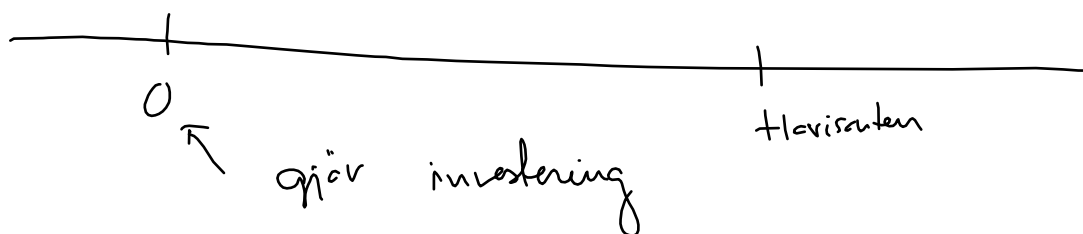


Aktyer og aukastning

Ser på et marked med d forskellige aktyer.

T_0 tidspunkt



Tidspunkt 0: Aktyerpriser: $S_{1,0}, S_{2,0}, \dots, S_{d,0}$

På vektorform $\vec{S}_0 = \begin{pmatrix} S_{1,0} \\ S_{2,0} \\ \vdots \\ S_{d,0} \end{pmatrix}$

Horisont: Aktyerpriser: $S_1, S_2, \dots, S_d$

$$\vec{S} = \begin{pmatrix} S_1 \\ S_2 \\ \vdots \\ S_d \end{pmatrix}$$

Investering: a_1 aktyer af type 1, a_2 aktyer af type 2, ... (kan være negative - short selling).
Vektorform $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{pmatrix}$

Verdi: Ved tid 0: $W_0 = a_1 S_{1,0} + a_2 S_{2,0} + \dots + a_d S_{d,0}$
 $= \vec{a}^T \vec{S}_0$

Ved horisonten: $W = a_1 S_1 + a_2 S_2 + \dots + a_d S_d = \vec{a}^T \vec{S}$

Aukastning for aktye i : $R_i = \frac{S_i - S_{i,0}}{S_{i,0}}$

Aukastning for porteføljen: $R = \frac{W - W_0}{W_0}$

Sammenheng: $R = \frac{\vec{a}^T \vec{S} - \vec{a}^T \vec{S}_0}{W_0} = \frac{\vec{a}^T (\vec{S} - \vec{S}_0)}{W_0}$
 $= \frac{\sum_{i=1}^d a_i (S_i - S_{i,0})}{W_0} = \sum_{i=1}^d \frac{a_i S_{i,0}}{W_0} \left(\frac{S_i - S_{i,0}}{S_{i,0}} \right)$

$$= \sum_{i=1}^d \underbrace{\frac{a_i S_{i,0}}{W_0}}_{w_i} R_i = \sum_{i=1}^d w_i R_i = \vec{w}^T \vec{R}$$

Her $w_i = \frac{a_i S_{i,0}}{W_0}$ og $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_d \end{pmatrix}$

Tolkning: w_i er andelen investert i aktye i .

(Test: $\sum_{i=1}^d w_i = \sum_{i=1}^d \frac{a_i S_{i,0}}{W_0} = \frac{1}{W_0} \underbrace{\sum_{i=1}^d a_i S_{i,0}}_{W_0} = \frac{1}{W_0} W_0 = 1$)

Hvordan ser risikoprofilen ut ved tiden 0?

Vi kjenner $S_{i,0}$, men ikke S_i .

Aukastingerne: R_i vil være ulikhet, men vi kan prøve å
estimere dem og sammenhengen fra historiske data.

Dette gir oss et estimat for gjennomsnitt $\bar{r}_i$ og
kovariansmatrisen C .

Hva sier dette om aukastningen på portefoljen:

$$R = \vec{w}^T \vec{R} = \sum_{i=1}^d w_i R_i \quad ?$$

$$\text{Da har vi} \quad \mu_R = E[R] = E\left[\sum_{i=1}^d w_i R_i\right] = \sum_{i=1}^d w_i E[R_i] = \sum_{i=1}^d w_i \bar{r}_i$$

$$\begin{aligned} \sigma_R^2 &= E[(R - \mu_R)^2] = E\left[\left(\sum_{i=1}^d w_i R_i - \sum_{i=1}^d w_i \bar{r}_i\right)^2\right] \\ &= E\left[\left(\sum_{i=1}^d w_i (R_i - \bar{r}_i)\right)^2\right] \end{aligned}$$

$$= E\left[\sum_{i=1}^d w_i (R_i - \bar{r}_i) = \sum_{j=1}^d w_j (R_j - \bar{r}_j)\right]$$

$$= E\left[\sum_{i,j=1}^d w_i (R_i - \bar{r}_i) w_j (R_j - \bar{r}_j)\right]$$

$$= \sum_{i,j=1}^d w_i w_j \underbrace{E[(R_i - \bar{r}_i)(R_j - \bar{r}_j)]}_{C_{ij}}$$

$$= \sum_{i,j=1}^d w_i w_j C_{ij} = \vec{w}^T C \vec{w}$$

Oppsummering: Formulering til portefolje aukastning

$$\mu_R = \vec{w}^T \vec{r} \quad \longleftarrow$$

$$\sigma_R^2 = \vec{w}^T C \vec{w} \quad \longleftarrow$$

Oppgaver:

1a) (side 151) Maks/min

$$f(x,y) = 4x - 3y$$

under betingelsen $\underbrace{x^2 + y^2 = 1}_{g(x,y)}$

$$\nabla f = (4, -3), \quad \nabla g = (2x, 2y)$$

$$\left. \begin{array}{l} 4 = 2x \\ -3 = 2y \\ x^2 + y^2 = 1 \end{array} \right\} \Rightarrow -\frac{4}{3} = \frac{x}{y} \Rightarrow \underline{\underline{x = -\frac{4}{3}y}}$$

Setter inn i andre ligning

$$\left(-\frac{4}{3}y\right)^2 + y^2 = 1$$

$$\frac{16}{9}y^2 + \frac{9}{9}y^2 = 1$$

$$\frac{25}{9}y^2 = 1$$

$$y^2 = \frac{9}{25} \Rightarrow y = \pm \frac{3}{5}$$

Hvis $y = \frac{3}{5}$, da er $x = -\frac{4}{3} \cdot \frac{3}{5} = -\frac{4}{5}$. Et punkt er $(-\frac{4}{5}, \frac{3}{5})$

Hvis $y = -\frac{3}{5}$, da er $x = -\frac{4}{3}(-\frac{3}{5}) = \frac{4}{5}$, et annet punkt $(\frac{4}{5}, -\frac{3}{5})$

Setter inn i f :

$$(-\frac{4}{5}, \frac{3}{5}): f(-\frac{4}{5}, \frac{3}{5}) = 4(-\frac{4}{5}) - 3(\frac{3}{5}) = -\frac{16}{5} - \frac{9}{5} = -\frac{25}{5} = \underline{\underline{-5}}$$

$$(\frac{4}{5}, -\frac{3}{5}): f(\frac{4}{5}, -\frac{3}{5}) = 4(\frac{4}{5}) - 3(-\frac{3}{5}) = \frac{16}{5} + \frac{9}{5} = 5 \text{ — maksimum.}$$

minimum



Oppgave 1d) Finn maks/min til

$$f(x,y,z) = x+y+z$$

$$\text{med } \underbrace{x^2+y^2=1}_{g(x,y,z)}, \quad \underbrace{2x+z=1}_{h(x,y,z)} \leftarrow z = 1-2x$$

Gradienten: $\nabla f = (1, 1, 1)$

$$\nabla g = (2x, 2y, 0)$$

$$\nabla h = (2, 0, 1)$$

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

Ligning: $1 = 2\lambda x + 2\mu = 2\lambda x + 2\mu$

$$1 = 2\lambda y + 0 = 2\lambda y$$

$$1 = 2 \cdot 0 + \mu \cdot 1 = \mu \Rightarrow \mu = 1$$

$$1 = 2\lambda x + 2 \Rightarrow 2\lambda x = -1$$

$$\frac{1}{-1} = \frac{2\lambda y}{2\lambda x} = \frac{y}{x} \Rightarrow \underline{y = -x}$$

Følger randbetingelsen $x^2+y^2=1$: $x^2+(-x)^2=1 \Rightarrow 2x^2=1$

$$\Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm \frac{1}{\sqrt{2}} = \pm \frac{\sqrt{2}}{2}$$

Hvis $x = \frac{\sqrt{2}}{2}$, da er $y = -\frac{\sqrt{2}}{2}$, $z = 1 - 2 \cdot \frac{\sqrt{2}}{2} = 1 - \sqrt{2}$: $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1 - \sqrt{2})$

Hvis $x = -\frac{\sqrt{2}}{2}$, da er $y = \frac{\sqrt{2}}{2}$, $z = 1 - 2(-\frac{\sqrt{2}}{2}) = 1 + \sqrt{2}$:

$$(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1 + \sqrt{2})$$

Setter inn i $f(x,y,z) = x+y+z$:

$$(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1 - \sqrt{2}) : f(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1 - \sqrt{2}) = 1 - \sqrt{2} \leftarrow \text{min}$$

$$(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1 + \sqrt{2}) : f(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1 + \sqrt{2}) = 1 + \sqrt{2} \leftarrow \text{maks}$$