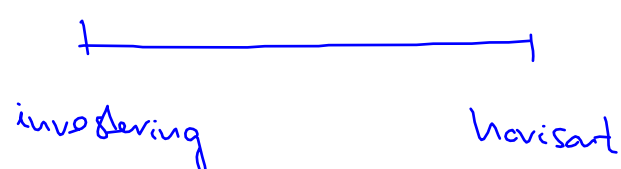


Førrige gang:



d aksjer: $1, 2, \dots, d$ med avkastning $R_1, R_2, \dots, R_d$

Investering: Andeler $w_1, w_2, \dots, w_d$, $\sum_{i=1}^d w_i = 1$ (Ikke negative w_i)

Forventet avkastning: $r_1, r_2, \dots, r_d$: $\vec{r} = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_d \end{pmatrix}$

Kovarians: $c_{ij} = \text{cov}(R_i, R_j)$,

$$C = \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1d} \\ \vdots & \vdots & \ddots & \vdots \\ c_{d1} & c_{d2} & \dots & c_{dd} \end{pmatrix} \text{ symmetrisk}$$

Forventning til porteføljen: $\sum_{i=1}^d w_i r_i = \vec{w}^T \vec{r}$

Varianser: $\sigma_R^2 = \vec{w}^T C \vec{w}$

Eksempel Anta vi har 2 aksjer, $d=2$ med avkastninger r_1, r_2 , $r_1 \neq r_2$. Vi ønsker å lage en portefølje slik at avkastningen blir r .

Ligninger:
$$\begin{array}{l} w_1 r_1 + w_2 r_2 = r \\ w_1 + w_2 = 1 \end{array} \quad \left| \begin{array}{l} \\ -r_2 \end{array} \right.$$

$$w_1 r_1 - w_1 r_2 = r - r_2 \Rightarrow w_1 = \frac{r - r_2}{r_1 - r_2}, w_2 = \frac{r_1 - r}{r_1 - r_2}$$

Sætning: Antag at C er en symmetrisk og invertibel matrix. Da

(i) C^{-1} er også symmetrisk

(ii) For alle søjle vektorer $\vec{x}, \vec{y}$ er

$$\vec{x}^T C^{-1} \vec{y} = \vec{y}^T C^{-1} \vec{x}$$

Bevis: (i) $(C^{-1})^T = (C^T)^{-1} = C^{-1}$

(ii) Siden $\vec{x}^T C^{-1} \vec{y}$ er en 1×1 -matrix, så

$$(\vec{x}^T C^{-1} \vec{y}) = (\vec{x}^T C^{-1} \vec{y})^T = \vec{y}^T (C^{-1})^T (\vec{x}^T)^T = \vec{y}^T C^{-1} \vec{x}$$

Problembeskrivelse: Ønsker en given forventet afkastning r med mindst mulig risiko.

Mathematisk: Vil minimere

$$\frac{1}{2} \vec{w}^T C \vec{w} \quad \text{under betingelserne}$$

$$\vec{w}^T \vec{r} = r \quad \text{og} \quad \sum w_i = 1, \text{ dvs. } \vec{w}^T \vec{1} = 1 \quad \text{der}$$

$$\vec{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

Omformning:

$$f(w_1, \dots, w_d) = \frac{1}{2} \sum_{i,j=1}^d C_{ij} w_i w_j$$

$$g(w_1, \dots, w_d) = r_1 w_1 + r_2 w_2 + \dots + r_d w_d = r$$

$$h(w_1, \dots, w_d) = w_1 + w_2 + \dots + w_d = 1$$

Lagrange: $\nabla f(\vec{w}) = \lambda \nabla g(\vec{w}) + \mu \nabla h(\vec{w})$

$$\text{dvs.} \quad \vec{w}^T \vec{r} = r, \quad \vec{w}^T \vec{1} = 1$$

Gradienterne: $\frac{\partial f}{\partial w_i} = r_i, \quad \nabla g = \vec{r}$

$$\frac{\partial h}{\partial w_i} = 1, \quad \nabla h = \vec{1}$$

$$\frac{\partial f}{\partial w_i} = \frac{1}{2} \frac{\partial}{\partial w_i} \left(\sum_{j=1}^d C_{ij} w_i w_j \right) = \frac{1}{2} \left(\sum_{j \neq i} 2 C_{ij} w_j + 2 C_{ii} w_i \right)$$

$$= \sum_{j=1}^d C_{ij} w_j \quad \begin{array}{l} \text{i-te ræk i matrix} \\ C \text{ med } \vec{w} \end{array}$$

$$\nabla f = C \vec{w}$$

Tilføjer til Lagrange: $C \vec{w} = \lambda \vec{r} + \mu \vec{1}$ | gangvis med C^{-1} fra venstre.

$$\vec{w} = C^{-1} C \vec{w} = \lambda C^{-1} \vec{r} + \mu C^{-1} \vec{1}$$

For at finde λ og μ , indsætter vi dette ind i betingelserne

$$\vec{w}^T \vec{r} = r, \quad \vec{w}^T \vec{1} = 1$$

$$\begin{aligned} \text{Begynder først ud } \vec{w}^T &= (\lambda \vec{r}^T + \mu \vec{1}^T)^T = \lambda (\vec{r}^T C^{-1}) + \mu (\vec{1}^T C^{-1}) \\ &= \lambda \vec{r}^T C^{-1} + \mu \vec{1}^T C^{-1} \quad (\text{under d. } (C^{-1})^T = C^{-1}) \end{aligned}$$

$$\text{Dermed: } \vec{w}^T \vec{r} = (\lambda \vec{r}^T C^{-1} + \mu \vec{1}^T C^{-1}) \vec{r}$$

$$= \lambda \underbrace{\vec{r}^T C^{-1} \vec{r}}_b + \mu \underbrace{\vec{1}^T C^{-1} \vec{r}}_a = r$$

$$\vec{w}^T \vec{1} = (\lambda \vec{r}^T C^{-1} + \mu \vec{1}^T C^{-1}) \vec{1}$$

$$= \lambda \underbrace{\vec{r}^T C^{-1} \vec{1}}_c + \mu \underbrace{\vec{1}^T C^{-1} \vec{1}}_d = 1$$

vedlægges

Nåværelse: $b = \vec{r}^T C^{-1} \vec{r}$

$$a = \vec{1}^T C^{-1} \vec{r} = \vec{r}^T C^{-1} \vec{1}$$

$$c = \vec{1}^T C^{-1} \vec{1}$$

Sætter ind:

$$\begin{array}{l|l|l} b\lambda + a\mu = r & c & a \\ a\lambda + c\mu = 1 & -a & -b \end{array}$$

$$cb\lambda - a^2\lambda = cr - a \Rightarrow \lambda = \frac{cr - a}{cb - a^2} = \frac{cr - a}{b(c - a^2)}$$

$$a^2\mu - bc\mu = ar - b \Rightarrow \mu = \frac{ar - b}{a^2 - bc} = \frac{b - ar}{b(c - a^2)}$$

Den optimale portefølge er

$$\vec{w}^* = \lambda C^{-1} \vec{r} + \mu C^{-1} \vec{1} = \frac{cr - a}{b(c - a^2)} C^{-1} \vec{r} + \frac{b - ar}{b(c - a^2)} C^{-1} \vec{1}$$

$$= \frac{b C^{-1} \vec{1} - a C^{-1} \vec{r}}{b(c - a^2)} + \frac{c C^{-1} \vec{r} - a C^{-1} \vec{1}}{b(c - a^2)} r$$

uafhængig af r

afhængig af r

Hva er risikoen til den optimale portefølje?

$$\sigma_x^2 = \underline{\vec{w}^*}^T \underline{C} \underline{\vec{w}^*}, \quad \underline{w^*} = \lambda \underline{C^{-1} \vec{r}} + \mu \underline{C^{-1} \vec{1}}$$

$$\underline{w^*}^T = \lambda \underline{\vec{r}^T C^{-1}} + \mu \underline{\vec{1}^T C^{-1}}$$

$$\sigma_x^2 = (\lambda \underline{\vec{r}^T C^{-1}} + \mu \underline{\vec{1}^T C^{-1}}) \cdot \underline{C} (\lambda \underline{C^{-1} \vec{r}} + \mu \underline{C^{-1} \vec{1}})$$

$$= (\lambda \underline{\vec{r}^T C^{-1}} + \mu \underline{\vec{1}^T C^{-1}}) (\lambda \underline{\vec{r}} + \mu \underline{\vec{1}})$$

$$= \lambda^2 \underbrace{\vec{r}^T C^{-1} \vec{r}}_b + \lambda \mu \underbrace{\vec{r}^T C^{-1} \vec{1}}_a + \lambda \mu \underbrace{\vec{1}^T C^{-1} \vec{r}}_a + \mu^2 \underbrace{\vec{1}^T C^{-1} \vec{1}}_c$$

$$= \lambda^2 b + 2\lambda\mu a + \mu^2 c \quad (\text{setter vi verdiene til } \lambda, \mu)$$

$$= \frac{(cr-a)^2 b + 2(cr-a)(b-ar) + (b-ar)^2 c}{(bc-a^2)^2} \quad \lambda = \frac{cr-a}{bc-a^2}, \quad \mu = \frac{b-ar}{bc-a^2}$$

$$= \frac{bc^2r^2 - 2abcr + a^2b + 2abcr - a^2cr^2 - a^2b + 2a^3r + b^2c - 2abcr + a^2c^2r^2}{(bc-a^2)^2}$$

$$= \frac{(bc^2 - a^2c)r^2 + 2(a^3 - abc)r + b^2c - a^2b}{(bc-a^2)^2}$$

$$= \frac{c(bc-a^2)r^2 + 2a(a^2-bc)r + b^2(bc-a^2)}{(bc-a^2)^2}$$

$$= \frac{cr^2 - 2ar + b}{bc-a^2} = \frac{c(r^2 - 2\frac{a}{c}r) + b}{bc-a^2}$$

$$= \frac{c(r^2 - 2\frac{a}{c}r + (\frac{a}{c})^2) - \frac{a^2}{c} + b}{bc-a^2}$$

$$= \frac{c(r - \frac{a}{c})^2 + \frac{1}{c}(bc-a^2)}{bc-a^2} = \frac{c(r - \frac{a}{c})^2}{bc-a^2} + \frac{1}{c}$$

Altså

$$\sigma_x^2 = \frac{c(r - \frac{a}{c})^2}{bc-a^2} + \frac{1}{c}$$

