Dynamiske systemer

Hva er et dynamisk system?

- Et sett av størrelser
- Tilstander = verdier på størrelsen
- En dynamikk som beskriver overgang mellom tilstander

Eksempel

1) Værvarsling:
   - størrelsene er trykk, temperatur, vind etc
   - dynamikken er beskrevet av naturlover dvs hvordan størrelsen endrer seg under gjensidig påvirkning

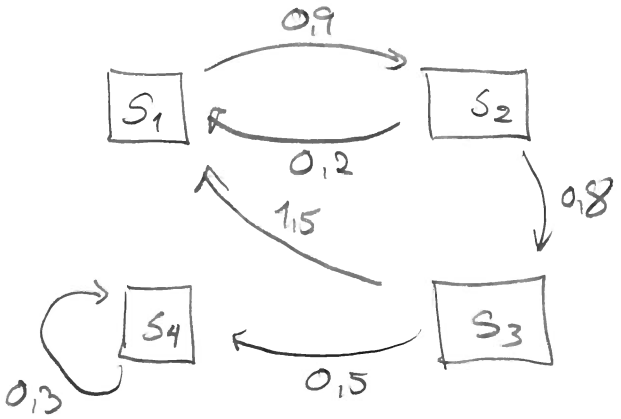
Et kontinuerlig dynamisk system

2) Befolkningsutvikling:
   - størrelsene en antall i hver aldersgruppe (n år)
   - dynamikk er hvordan antallene påvirker hverandre

Et diskret dynamisk system

3) Når værvarsling skal gjøres overføres et kontinuerlig system til et diskret for å tilpasses datamaskinens begrensede kapasitet.

Illustrasjon av et distret DgS:

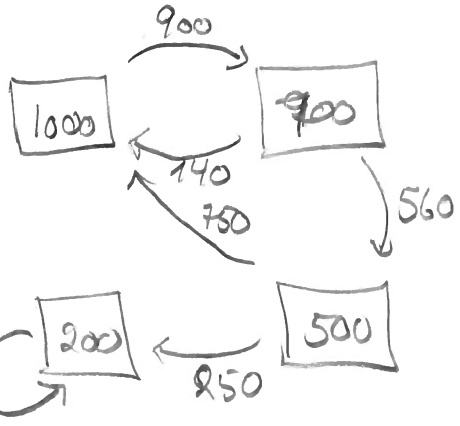


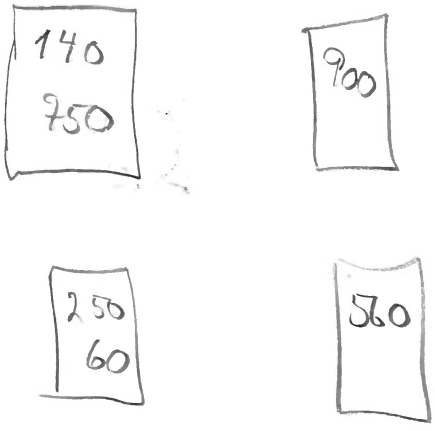
Populasjon:
- 4 klasser S_1, S_2, S_3, S_4
- Pilene beskriver dynamikken i ét generasjonsskritt
- Ingen tilstander angitt

Eksempel på tilstand (angis som søylevektor)

$$\bar{V}_0 = \begin{pmatrix} 1000 \\ 700 \\ 500 \\ 200 \end{pmatrix}$$

etter ett skritt $\bar{V}_1 = \begin{pmatrix} 890 \\ 900 \\ 560 \\ 310 \end{pmatrix}$





$$A = \begin{pmatrix} 0 & 0,2 & 1,5 & 0 \\ 0,9 & 0 & 0 & 0 \\ 0 & 0,8 & 0 & 0 \\ 0 & 0 & 0,5 & 0,3 \end{pmatrix}$$

Overgangsmatrise = Dynamikk

$A\bar{V}_0 = \bar{V}_1$

$$\bar{v}_2 = A\bar{v}_1 = AA\bar{v}_0 = A^2\bar{v}_0 = \begin{pmatrix} 1020 \\ 801 \\ 720 \\ 373 \end{pmatrix}$$

$$A^{10} \approx \begin{pmatrix} 0{,}23 & 1{,}01 & 1{,}93 & 0 \\ 1{,}16 & 0{,}23 & 0{,}85 & 0 \\ 0{,}45 & 1{,}03 & 0{,}11 & 0 \\ 0{,}18 & 0{,}31 & 0{,}71 & 0 \end{pmatrix}$$

$$\bar{v}_{10} = \begin{pmatrix} 1901 \\ 1741 \\ 1231 \\ 757 \end{pmatrix}$$

Summen av $\bar{v}_0$: 2400
" $\bar{v}_{10}$: 5630
" $\bar{v}_{11}$: 6139

Vi observerer $\dfrac{|\bar{v}_{n+1}|_1}{|\bar{v}_n|_1} \approx 1{,}084$

Anta at egenvektorene (for ulike egenverdier) er $\bar{w}_1, \ldots, \bar{w}_4$
Da har vi (største egenverdi)

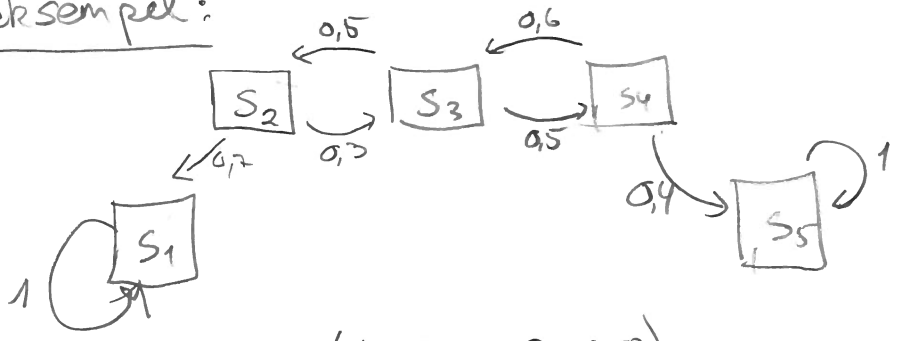
$$\bar{v}_0 = a\bar{w}_1 + b\bar{w}_2 + c\bar{w}_3 + d\bar{w}_4$$

$$\bar{v}_k = A^k \cdot \bar{v}_0 = a\lambda_1^k \bar{w}_1 + b\lambda_2^k \bar{w}_2 + c\lambda_3^k \bar{w}_3 + d\lambda_4^3$$

$\longrightarrow a\lambda_1^k \bar{w}_1$ blir nærmere 0.

I eksemplet er største egenverdi $\lambda_1 = 1{,}084$

Nytt eksempel:



$$A = \begin{pmatrix} 1 & 0{,}7 & 0 & 0 & 0 \\ 0 & 0 & 0{,}5 & 0 & 0 \\ 0 & 0{,}3 & 0 & 0{,}6 & 0 \\ 0 & 0 & 0{,}5 & 0 & 0 \\ 0 & 0 & 0 & 0{,}4 & 1 \end{pmatrix}$$

stokastisk matrise

$$\bar{v}_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \bar{v}_{100} = A^{100}\bar{v}_0 = \begin{pmatrix} 0{,}84 \\ 0 \\ 0 \\ 0 \\ 0{,}11 \end{pmatrix}$$

$$\chi_A(\lambda) = \det\begin{pmatrix} 1-\lambda & 0{,}7 & 0 & 0 & 0 \\ 0 & -\lambda & 0{,}5 & 0 & 0 \\ 0 & 0{,}3 & -\lambda & 0{,}6 & 0 \\ 0 & 0 & 0{,}5 & -\lambda & 0 \\ 0 & 0 & 0 & 0{,}4 & -\lambda \end{pmatrix}$$

$$= (1-\lambda)^2\left((-\lambda)^3 + 0{,}15\lambda + 0{,}3\lambda\right)$$

$$= (1-\lambda)^2\,\lambda\left(\lambda^2 - 0{,}45\right) = 0 \quad \text{betyr} \quad \lambda_1 = \lambda_2 = 1 \geq \lambda_3 = \sqrt{0{,}45} \geq 0 \geq -\sqrt{0{,}45}.$$

Egenvektor for $\lambda = 1$

$$\begin{pmatrix} 0 & 0{,}7 & 0 & 0 & 0 \\ 0 & -1 & 0{,}5 & 0 & 0 \\ 0 & 0{,}3 & -1 & 0{,}6 & 0 \\ 0 & 0 & 0{,}5 & -1 & 0 \\ 0 & 0 & 0 & 0{,}4 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0{,}7\cdot x_2 \\ -x_2 + 0{,}5x_3 \\ 0{,}3x_2 - x_3 + 0{,}6x_4 \\ 0{,}5x_3 - x_4 \\ 0{,}4x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad x_2 = 0 \quad x_3 = 0 \quad x_4 = 0.$$

dvs egenvektorer $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ og $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

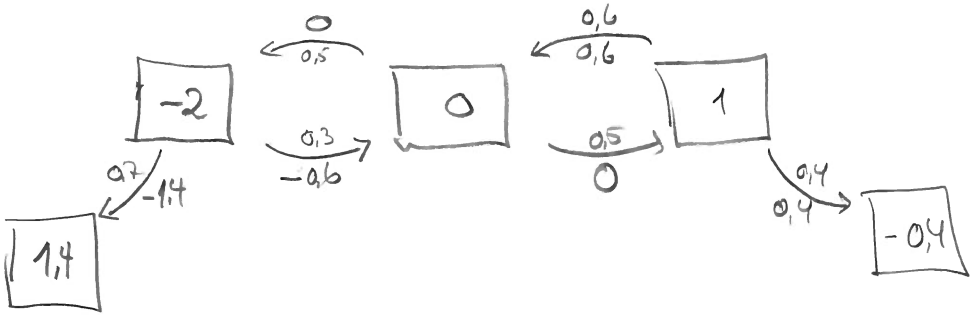
Egenvektor for $\lambda = 0$

$$\begin{pmatrix} 1 & 0{,}7 & 0 & 0 & 0 \\ 0 & 0 & 0{,}5 & 0 & 0 \\ 0 & 0{,}3 & 0 & 0{,}6 & 0 \\ 0 & 0 & 0{,}5 & 0 & 0 \\ 0 & 0 & 0 & 0{,}4 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_1 + 0{,}7x_2 \\ 0{,}5x_3 \\ 0{,}3x_2 + 0{,}6x_4 \\ 0{,}5x_3 \\ 0{,}4\cdot x_4 + x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

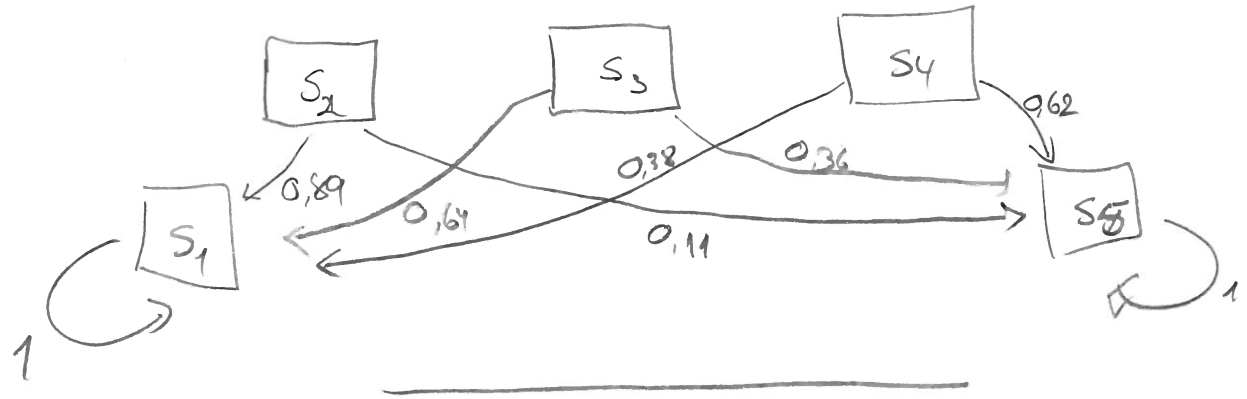
$$\Rightarrow \quad x_2 = -2x_4 \quad x_1 = -0{,}7x_2 = 1{,}4x_4$$

$$x_3 = 0 \quad x_5 = -0{,}4x_4 \text{ e}$$

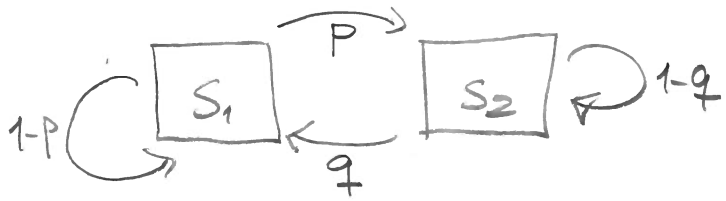
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_4\begin{pmatrix} 1{,}4 \\ -2 \\ 0 \\ 1 \\ -0{,}4 \end{pmatrix}$$



$$A^{100} = \begin{pmatrix} 1 & 0,89 & 0,64 & 0,38 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0,11 & 0,36 & 0,62 & 1 \end{pmatrix}$$



Eksempel



$$A = \begin{pmatrix} 1-p & q \\ p & 1-q \end{pmatrix}$$

272

$$\chi_A(\lambda) = \lambda^2 - (2-p-q)\lambda + \left((1-p)(1-q) - pq\right) = 0$$

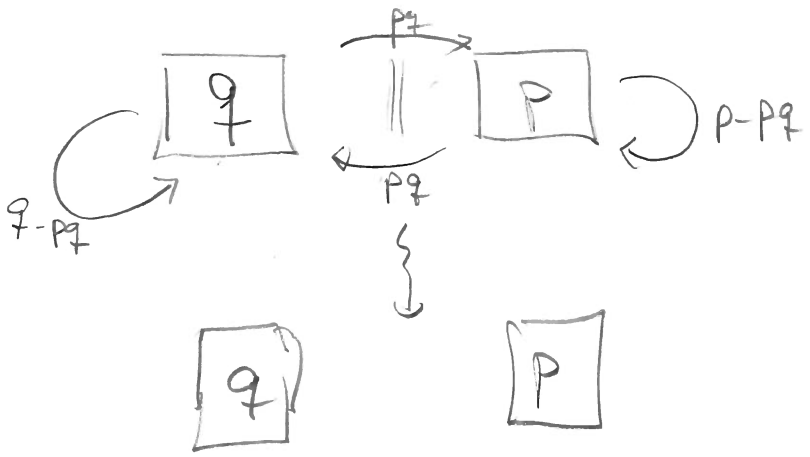
$$\lambda = \frac{2-p-q \pm \sqrt{(2-p-q)^2 - 4(1-p-q)}}{2}$$

$$= \frac{2-p-q \pm \sqrt{4 - 4(p+q) + (p+q)^2 - 4 + 4(p+q)}}{2}$$

$$= \frac{2-p-q \pm (p+q)}{2} = \begin{cases} 1 \\ 1-p-q \end{cases}$$

Egenvektor for $\lambda = 1$.

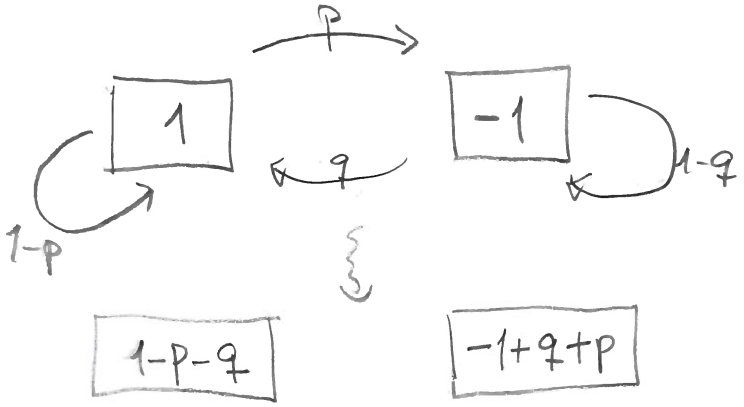
$$\begin{pmatrix} 1-p-1 & q \\ p & 1-q-1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -px+qy \\ px-qy \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} q \\ p \end{pmatrix}$$



Egenverdier for $1-p-q$:

$$\begin{pmatrix} 1-p-(1-p-q) & q \\ p & 1-q-(1-p-q) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} q & q \\ p & p \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad y = -x$$



$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{ap-bq}{p+q}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{a+b}{p+q}\begin{pmatrix} q \\ p \end{pmatrix} \xrightarrow{A^n} (1-p-q)^n \frac{ap-bq}{p+q}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{a+b}{p+q}\begin{pmatrix} q \\ p \end{pmatrix}$$

$$\xrightarrow{n\to\infty} \frac{a+b}{p+q}\begin{pmatrix} q \\ p \end{pmatrix}$$

$$0 \le p, q \le 1$$
$$\Updownarrow$$
$$0 \le p+q \le 2$$
$$\Updownarrow$$
$$-2 \le -p-q \le 0$$
$$\Updownarrow$$
$$-1 \le 1-p-q \le 1$$
$$\Updownarrow$$
$$|1-p-q| \le 1$$

Hvis vi starter med $a+b=1$ får vi $\longrightarrow \frac{1}{p+q}\begin{pmatrix} q \\ p \end{pmatrix}$

likevekts tilstand.

$1-p-q \approx 0$ Mye som skjer

$|1-p-q| \ll 1$ Lite som skjer