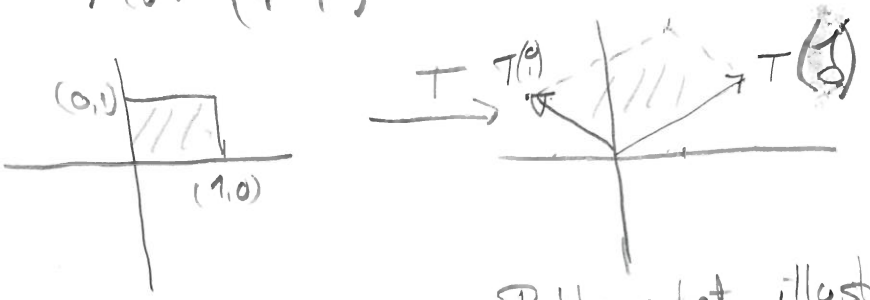
Geometrisk tolkning av lineære avbildninger

Skal se på $T: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad T(\bar{v}) = A \cdot \bar{v}$

↑ kvadratisk 2×2

Generell illustrasjon

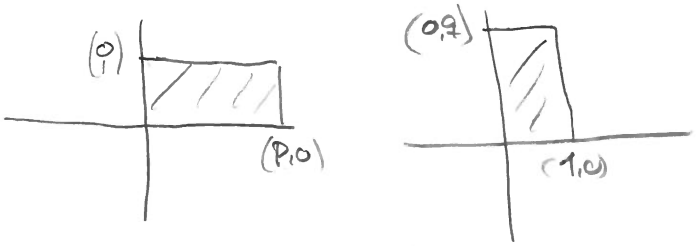
$$T(\bar{v}) = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \bar{v}$$



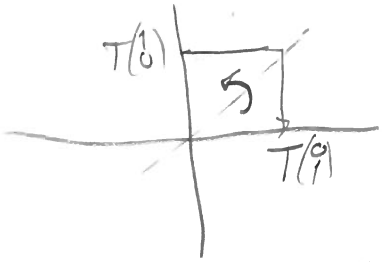
Rektangelet illustrere T.

Alle 2×2-matriser kan skrives som et produkt av elementære matriser (svarer til elementære radoperasjoner)

1) Skalering $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ eller $\begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}$

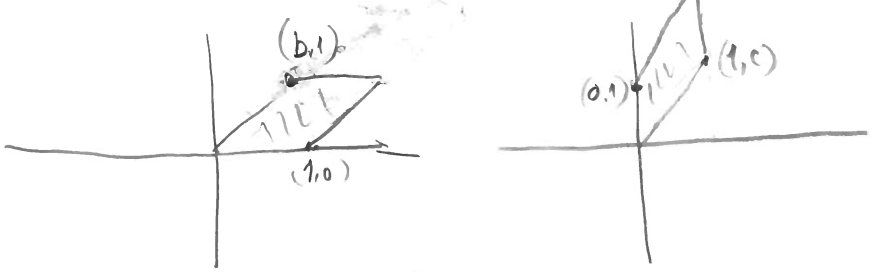


2) Permutering $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$



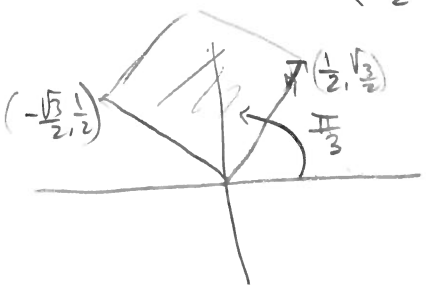
speiling om $x=y$

3) Forskyvning $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ eller $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$



Eksempel: Rotasjon med vinkel $\theta = \frac{\pi}{3}$.

gir rad $R_{\frac{\pi}{3}} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

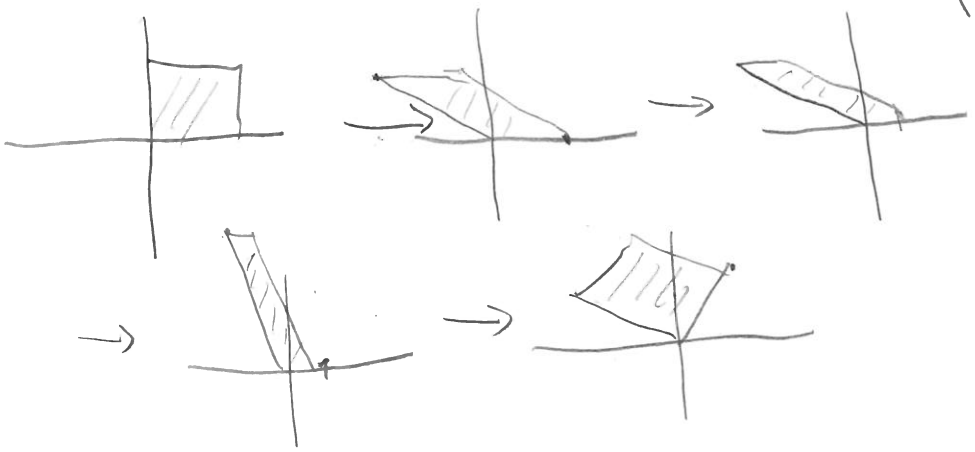


$$\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \xrightarrow{II - \sqrt{3} \cdot I} \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 2 \end{pmatrix} \xrightarrow{\frac{1}{2} II} \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 1 \end{pmatrix}$$

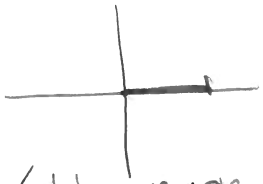
$$\xrightarrow{2 \cdot I} \begin{pmatrix} 1 & -\sqrt{3} \\ 0 & 1 \end{pmatrix} \xrightarrow{I + \sqrt{3} II} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \sqrt{3} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}}_{\begin{pmatrix} 1 & 0 \\ \sqrt{3} & 2 \end{pmatrix}} \underbrace{\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\sqrt{3} \\ 0 & 1 \end{pmatrix}}_{\begin{pmatrix} \frac{1}{2} & 0 \\ \frac{\sqrt{3}}{2} & 2 \end{pmatrix}} \underbrace{\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}}$$

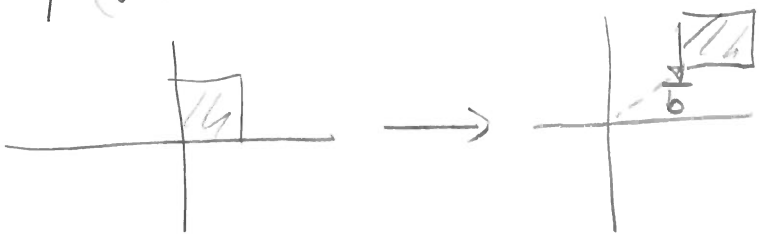


To andre tilfeller:

4) Projeksjon  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

5) Translasjon (ikke noen lineær avbildning)

$$T(\bar{v}) = \bar{v} + \bar{b}$$

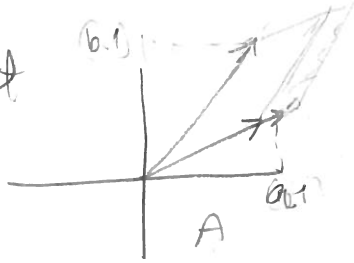


Symmetrisk matrise $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$

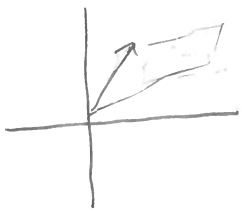


Produkt: sette sammen avbildninger

Transponert



$\begin{pmatrix} 1 & b \\ a & 1 \end{pmatrix}$



$\begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix}$

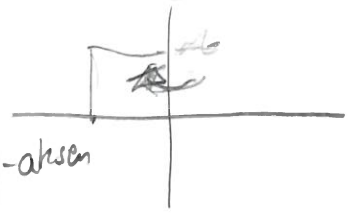
Ortogonal. To typer

Rotasjon $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

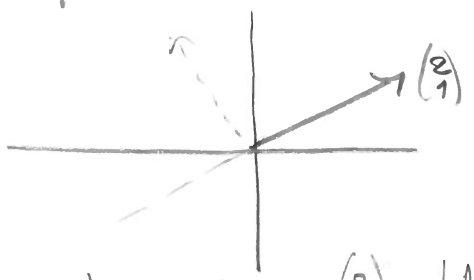


$\det(R_\theta) = \cos^2\theta + \sin^2\theta = 1$

Refleksjon $S = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$



Refleksjon om annen linje:



1) Roterer $\begin{pmatrix}2\\1\end{pmatrix}$ til $\begin{pmatrix}\sqrt{5}\\0\end{pmatrix}$: $\frac{1}{\sqrt{5}}\begin{pmatrix}2 & 1\\ -1 & 2\end{pmatrix}$

2) Speiler om x-aksen. $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

3) Roterer $\begin{pmatrix}\sqrt{5}\\0\end{pmatrix}$ til $\begin{pmatrix}2\\1\end{pmatrix}$: $\frac{1}{\sqrt{5}}\begin{pmatrix}2 & -1\\ 1 & 2\end{pmatrix}$

$$\frac{1}{\sqrt{5}}\begin{pmatrix}2 & -1\\ 1 & 2\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix}2 & 1\\ -1 & 2\end{pmatrix} = \frac{1}{5}\begin{pmatrix}2 & 1\\ 1 & -2\end{pmatrix}\begin{pmatrix}2 & 1\\ -1 & 2\end{pmatrix} = \frac{1}{5}\begin{pmatrix}3 & 4\\ 4 & -3\end{pmatrix}$$

Merk at rotasjon skal bevare 90°-vinkel, det betyr at når $\begin{pmatrix}2\\1\end{pmatrix} \longmapsto \begin{pmatrix}\sqrt{5}\\0\end{pmatrix}$ må matrisen være ortogonal

$$\begin{pmatrix}a & -b\\ b & a\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}\sqrt{5}\\0\end{pmatrix}$$

$$a^2 + b^2 = 1$$

dvs $\left.\begin{aligned} 2a - b &= \sqrt{5}\\ 2b + a &= 0\end{aligned}\right\}$ $\begin{pmatrix}a\\b\end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix}2\\-1\end{pmatrix}$

$$\frac{1}{5}\begin{pmatrix}3 & 4\\ 4 & -3\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix} \qquad \frac{1}{5}\begin{pmatrix}3 & 4\\ 4 & -3\end{pmatrix}\begin{pmatrix}-1\\2\end{pmatrix} = \begin{pmatrix}1\\-2\end{pmatrix}$$

Kan lage refleksjonsmatrise på en annen måte

Householdermatrise $\bar{v} = \begin{pmatrix}1\\-2\end{pmatrix}$

$$Q_{\bar{v}} = I - 2\frac{\bar{v}\cdot\bar{v}^T}{\bar{v}^T\bar{v}}$$

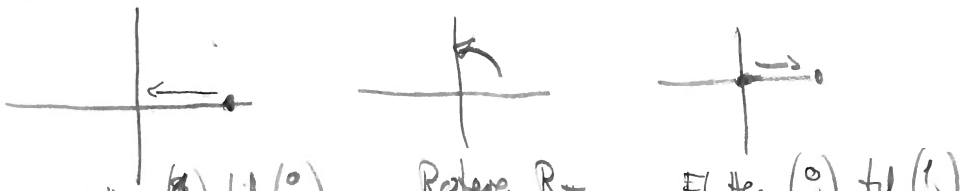
$\begin{pmatrix}1\\-2\end{pmatrix}(1\ -2) = \begin{pmatrix}1 & -2\\ -2 & 4\end{pmatrix}$

$(1\ -2)\begin{pmatrix}1\\-2\end{pmatrix} = 5$

$$= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - 2\,\frac{1}{5}\begin{pmatrix}1 & -2\\ -2 & 4\end{pmatrix} = \frac{1}{5}\begin{pmatrix}5 & 0\\ 0 & 5\end{pmatrix} - \frac{1}{5}\begin{pmatrix}2 & -4\\ -4 & 8\end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix}3 & 4\\ 4 & -3\end{pmatrix}$$

Rotasjon om punktet $\begin{pmatrix}1\\0\end{pmatrix}$ med vinkel $\frac{\pi}{3}$



Flytter $\begin{pmatrix}1\\0\end{pmatrix}$ til $\begin{pmatrix}0\\0\end{pmatrix}$ | Rotere $R_{\frac{\pi}{3}}$ | Flytter $\begin{pmatrix}0\\0\end{pmatrix}$ til $\begin{pmatrix}1\\0\end{pmatrix}$

$$R_{\frac{\pi}{3}}\left(\bar{v}-\begin{pmatrix}1\\0\end{pmatrix}\right)+\begin{pmatrix}1\\0\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1&-\sqrt{3}\\\sqrt{3}&1\end{pmatrix}\left(\bar{v}-\begin{pmatrix}1\\0\end{pmatrix}\right)+\begin{pmatrix}1\\0\end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix}1&-\sqrt{3}\\\sqrt{3}&1\end{pmatrix}\bar{v}-\frac{1}{2}\begin{pmatrix}1\\\sqrt{3}\end{pmatrix}+\begin{pmatrix}1\\0\end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix}1&-\sqrt{3}\\\sqrt{3}&1\end{pmatrix}\bar{v}+\frac{1}{2}\begin{pmatrix}1\\-\sqrt{3}\end{pmatrix}$$

Merk $\quad\frac{1}{2}\begin{pmatrix}1&-\sqrt{3}\\\sqrt{3}&1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}+\frac{1}{2}\begin{pmatrix}1\\-\sqrt{3}\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1\\\sqrt{3}\end{pmatrix}+\frac{1}{2}\begin{pmatrix}1\\-\sqrt{3}\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$

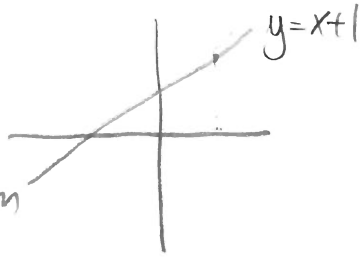
Rotasjoner fikserer rotasjonssenter

Refleksjoner fikserer refleksjonsakse

$$\frac{1}{5}\begin{pmatrix}3&4\\4&-3\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix}$$

$$\frac{1}{5}\begin{pmatrix}3&4\\4&-3\end{pmatrix}\begin{pmatrix}1\\-2\end{pmatrix}=\begin{pmatrix}-1\\2\end{pmatrix}$$

Refleksjon om linja $x-y=-1$



1) Flytt linja så den går gjennom origo $\quad T(\bar{x})=\bar{x}-\begin{pmatrix}0\\1\end{pmatrix}$
2) Speiler om linja $y=x$: $\quad\begin{pmatrix}0&1\\1&0\end{pmatrix}$
3) Flytt tilbake $\quad\bar{x}\mapsto\bar{x}+\begin{pmatrix}0\\1\end{pmatrix}$

$$\begin{pmatrix}0&1\\1&0\end{pmatrix}\left(\bar{x}-\begin{pmatrix}0\\1\end{pmatrix}\right)+\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0&1\\1&0\end{pmatrix}\bar{x}+\begin{pmatrix}-1\\1\end{pmatrix}$$

Sjekk $\quad\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix}+\begin{pmatrix}-1\\1\end{pmatrix}=\underline{\begin{pmatrix}1\\2\end{pmatrix}}$

og $\quad\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix}+\begin{pmatrix}-1\\1\end{pmatrix}=\underline{\underline{\begin{pmatrix}-1\\1\end{pmatrix}}}\qquad\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}-1\\0\end{pmatrix}+\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}0\\-1\end{pmatrix}+\begin{pmatrix}-1\\1\end{pmatrix}=\underline{\underline{\begin{pmatrix}-1\\0\end{pmatrix}}}$