Geometri i rommet

Enkle generalisering av $\mathbb{R}^2$	Spesielle egenskaper ved $\mathbb{R}^3$

Fakta

Lengde $\|(a,b,c)\| = \sqrt{a^2+b^2+c^2}$

Ortogonalitet $\bar{v}^T\bar{w} = 0$

Kryssprodukt finnes ikke i $\mathbb{R}^2$

$$\bar{v} \times \bar{w} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Kalles produkt "fordi"

$$\bar{u} \times (\bar{v} + \bar{w}) = \bar{u} \times \bar{v} + \bar{u} \times \bar{w}$$

Merk (siden dette er en determinant)

$$\bar{v} \times \bar{w} = -\bar{w} \times \bar{v}$$

$$\bar{v} \times \bar{v} = 0.$$

Betyr at alt er bestemt av $\bar{e}_i \times \bar{e}_j$ (eller $\bar{i}, \bar{j}, \bar{k}$)

$$\bar{i} \times \bar{j} = \bar{k}$$
$$\bar{j} \times \bar{k} = \bar{i}$$
$$\bar{k} \times \bar{i} = \bar{j}$$

Vi har også (Jacobi-identitet)

$$\bar{u} \times (\bar{v} \times \bar{w}) + \bar{v} \times (\bar{w} \times \bar{u}) + \bar{w} \times (\bar{u} \times \bar{v}) = \bar{0}$$

~~$\bar{e}_i \times \bar{e}_j$~~ $\bar{i} \times (\bar{j} \times \bar{k}) + \bar{j} \times (\bar{k} \times \bar{i}) + \bar{k} \times (\bar{i} \times \bar{j})$

$= \bar{i} \times \bar{i} + \bar{j} \times \bar{j} + \bar{k} \times \bar{k} = \bar{0}$.

$\bar{i} \times (\bar{j} \times \bar{j}) + \bar{j} \times (\bar{j} \times \bar{i}) + \bar{j} \times (\bar{i} \times \bar{j})$

$= \bar{0} + \bar{j} \times (-\bar{k}) + \bar{j} \times \bar{k}$

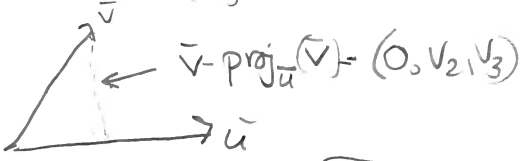
$= -\bar{i} + \bar{i} = \bar{0}$.

Anvendelse:

Produsere en vektor som står normalt på to gitte vektorer

Lengden av kryssproduktet er lik arealet av parallellogrammet

$$\bar{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \bar{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$



arealet $= 1 \cdot \sqrt{v_2^2 + v_3^2}$

$= \sqrt{v_2^2 + v_3^2}$

Kryssproduktet $\begin{pmatrix}1\\0\\0\end{pmatrix} \times \begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = \begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 1 & 0 & 0\\ v_1 & v_2 & v_3\end{vmatrix}$

$= \bar{k}\cdot v_2 - \bar{j}\cdot v_3 = \begin{pmatrix}0\\-v_3\\v_2\end{pmatrix}$

med lengde $\sqrt{v_2^2+v_3^2}$.

Kan brukes til å lage en ortogonal basis

$\bar{v}_1 = \begin{pmatrix}a\\b\\c\end{pmatrix}$ $\bar{v}_2 = \begin{pmatrix}-b\\a\\0\end{pmatrix}$ $\bar{v}_3 = \begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ a & b & c\\ -b & a & 0\end{vmatrix} = -bc\bar{j} + a^2\bar{k} + b^2\bar{k} - ac\bar{i}$

$= \begin{pmatrix}-ac\\-bc\\a^2+b^2\end{pmatrix}$

$\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ ortogonal

Tripelprodukt

$[\bar{u}, \bar{v}, \bar{w}] = \bar{u}\cdot(\bar{v}\times\bar{w}) = \begin{vmatrix}u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{vmatrix} = (\bar{u}\times\bar{v})\cdot\bar{w}$

gir volumet av parallellepiped

ortonormal basis:

$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\-2\end{pmatrix}$

$\frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}}\frac{1}{\sqrt{6}}\begin{vmatrix}1 & -1 & 0\\ 1 & 1 & 1\\ 1 & 1 & -2\end{vmatrix} = \frac{1}{6}(-2-1+0-0-2-1) = -1$

Interessant ulikhet

$[A\bar{u}, A\bar{v}, A\bar{w}] = \det(A)\,[\bar{u}, \bar{v}, \bar{w}]$.

$$[A\bar{u}, A\bar{v}, A\bar{w}] = \det\begin{pmatrix} A\bar{u}^T \\ A\bar{v}^T \\ A\bar{w}^T \end{pmatrix} = \det\left((A\bar{u})^* \mid (A\bar{v})^* \mid (A\bar{w})^*\right)$$

$$= \det(A \cdot (\bar{u} \mid \bar{v} \mid \bar{w}))$$
$$= \det(A) \cdot \det(\bar{u} \mid \bar{v} \mid \bar{w}) = \det(A)\,[\bar{u}, \bar{v}, \bar{w}]$$

Regne ut noen kryssprodukt

$$\bar{v} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \quad \bar{w} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \quad \bar{v} \times \bar{w} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 1 & -1 \\ 1 & 3 & 1 \end{vmatrix}$$

$$= \bar{i} - \bar{j} + 6\bar{k} - \bar{k} - 2\bar{j} + 3\bar{i}$$
$$= 4\bar{i} - 3\bar{j} + 5\bar{k} = \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix}$$

Alternativt:

$$(2\bar{i} + \bar{j} - \bar{k}) \times (\bar{i} + 3\bar{j} + \bar{k})$$

$$= 2\bar{i} \times \bar{i} + 6 \times \bar{i} \times \bar{j} + 2\,\bar{i} \times \bar{k}$$
$$+ \bar{j} \times \bar{i} + 3\,\bar{j} \times \bar{j} + \bar{j} * \bar{k}$$
$$- \bar{k} \times \bar{i} - 3\bar{k} \times \bar{j} - \bar{k} \times \bar{k}$$
$$= 6\bar{k} - 2\bar{j} - \bar{k} + \bar{i} - \bar{j} + 3\bar{i} = 4\bar{i} - 3\bar{j} + 5\bar{k}$$